AFRL-PR-WP-TR-1998-2108

DESCRIPTION OF THE THERMAL SPIKE RESULTING FROM THE EXPANSION OF A PERFECT GAS FROM AN INFINITE RESERVOIR INTO A VOID MAIN

Robert E. Gray

Turbine Branch
Turbine Engine Division
Propulsion Directorate
Wright-Patterson AFB OH 45324-7251

AUGUST 1998

FINAL REPORT FOR PERIOD 5 AUGUST 1997 – 1 AUGUST 1998

PROPULSION DIRECTORATE
AIR FORCE RESEARCH LABORATORY
AIR FORCE MATERIEL COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OH 45433-7251

NOTICE

This technical report has been reviewed and is approved for publication.

Robert E. Gray

ROBERT E. GRAY
Project Engineer
Trubine Branch
Turbine Engine Division
Propulsion Directorate

Charles D. MacArthur

CHARLES D. MacARTHUR
Chief, Turbine Branch
Turbine Engine Division
Propulsion Directorate

William E. Koop

WILLIAM E. KOOP
Chief of Technology
Turbine Engine Division
Propulsion Directorate

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	10 Aug 98	Final Report for Period 05 Aug 97 - 01 Aug 98

4. TITLE AND SUBTITLE

Description Of The Thermal Spike Resulting From The Expansion Of A Perfect Gas From An Infinite Reservoir Into A Void Main

5. FUNDING NUMBERS

C: None
PE: 62203F
PR: 3066
TA: 06
WU: 47

6. AUTHOR(S)

Robert E. Gray

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Turbine Branch
Turbine Engine Division
Propulsion Directorate
Wright-Patterson AFB OH 45324-7251

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Propulsion Directorate
Air Force Research Laboratory
Air Force Materiel Command
Wright-Patterson Air Force Base, OH 45433-7251
POC: Robert E. Gray, AFRL/PRTT, 937-255-3150

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

AFRL-PR-WP-TR-1998-2108

11. SUPPLEMENTARY NOTES

Report prepared internally by Sponsoring Agency

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The one-dimensional, inviscid equations of motion are applied to the problem of describing the shock and rarefaction wave which form when the flow of a perfect gas from an infinite reservoir into a vacuous main is initiated. Characteristic theory is applied to show that the rarefaction wave is a centered, simple, backward-facing expansion wave. It is further shown that the stagnation temperature is a strict function of the local sound speed in the domain of this wave, and that the sound speed has a fixed, distinct value on each of the straight characteristics of the wave. The Rankine-Hugoniot Relations are used to show that the shock coincides with the greatest lower bound of characteristics. The local sound speed monotonically decreases to zero as the limit of characteristics in the wave domain is approached, resulting in the stagnation temperature monotonically approaching its maximum value there. It is deduced that the stagnation temperature function described in this report represents the thermal spike observed during the establishment of flow in a short-duration test facility. It is concluded that the thermal spike is the artifact of the flow work done to form the shock at the initiation of the blow-down process in such a facility.

14. SUBJECT TERMS	15. NUMBER OF PAGES
Rarefaction, Simple Wave, Shock Wave, Stagnation Temperature, Rankine-Hugoniot Conditions	85
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF SYMBOLS

V = control volume (See context).
V = neighborhood of a point in range R of a function (See context).
V_{cv} = velocity of control volume enclosing a shock.
$\vec{V}$ = velocity vector at a point in a flow field.
ε = sub-volume of a control volume.
σ = control volume surface area (See context. Also, see Appendix).
$\hat{n}$ = unit normal to a surface.
$\mathscr{D}$ = domain of a function.
R = range of a function (See context).
R = universal gas constant divided by molecular weight (See context).
R = subscript for right face of control volume (See context).
L = subscript for left face of control volume (See context. See Appendix for other usage).
E = total energy of a fixed, finite amount of mass or matter.
Q = heat entrained by a fixed, finite amount mass or matter.
W_{cv} = work in all forms done by a thermodynamic system upon its environment.
cv = reference to a control volume when appearing as a subscript.
m = fixed, finite amount of fluid mass (See context).
m = distance between two points in the domain $\mathscr{D}$ of a function (See context).
m = mass flux at face of control volume V (See context).
p = local pressure existing at a point in a fluid medium (See context).
p = a particular point in domain $\mathscr{D}$ of a function signifying the intersection of characteristic curves (See context).
$\bar{p}$ = image in range R of point p in domain $\mathscr{D}$ of a function.
p_0 = absolute stagnation pressure at the plane of choked flow in the domain of the expansion wave.
p_1 = absolute static pressure at the plane of choked flow.
p_{R0} = absolute pressure at a point in the quiescent, or stagnant flow, interior to the infinite reservoir.
p_{cr} = static pressure at critical velocity.
v = specific volume of an element of fluid mass.
ρ = density of an element of fluid mass $\left(\rho = \dfrac{1}{v}\right)$.
ρ_1 = gas density at the plane of choked flow.
ρ_{R0} = gas density in the quiescent, or stagnant flow, interior to the infinite reservoir.
ϵ = real number greater than zero.

LIST OF SYMBOLS (continued)

δ = finite increment of a quantity (See context).

$\delta(\epsilon)$ = a real number greater than zero depending on ϵ (See context).

u = specific internal energy (See context).
u = point in domain $\mathscr{D}$ of a function (See context).
e = specific total energy (See context).
e = ordered pair of numbers specifying direction in the domain $\mathscr{D}$ of a function (See context).
h = ordered pair of numbers specifying direction in the range R of a function (See context).
h = total, or stagnation specific enthalpy (See context. See Appendix for usage).
t = time or time coordinate in x-t space.
x = x coordinate in x-t space (See context).
x = point in domain $\mathscr{D}$ of a function (See context).
s = specific entropy.
T = absolute temperature.

T_0 = absolute stagnation temperature at the plane of choked flow in the domain of the expansion wave.

T_1 = absolute static pressure at the plane of choked flow.

T_{R0} = absolute temperature at a point in the quiescent, or stagnant flow, interior to the infinite reservoir.

T_s = absolute stagnation temperature at a point in the domain of the expansion wave.

T_s^* = supremum of absolute stagnation temperature.

T_{s*} = infimum of absolute stagnation temperature.

T_z = absolute static temperature at a point in the domain of the expansion wave.

γ = ratio of specific heat at constant pressure to specific heat at constant volume for a calorically perfect gas.

α = neighborhood of a point z in a control volume.

$\frac{D}{Dt}$ = substantial derivative.

∇ = gradient symbol.

Λ = dissipation force per unit volume in a flowing fluid.

C = local sound speed in a medium.

q = distance between two points in the range of a function.
c = point in domain $\mathscr{D}$ of a function.

LIST OF SYMBOLS (continued)

d = point in the range R of a function.
y = point in the range R of a function.
z = point in the range R of a function (See context).
z = arbitrary pressure ratio (See context. See Appendix for usage).

$Df(c)$ = derivative of function f at point c.

U = x-axis component of fluid velocity (See context).

U = neighborhood of a point c in the domain $\mathscr{D}$ of a function (See context).

U_{cr} = critical, or choke velocity.

U_{l0} = limit velocity at the plane of choked flow.

U_L = escape velocity from expansion wave.

$\exists A$ = existential quantifier. (There exists A, . . .).

$\forall A$ = universal quantifier. (For all A, . . .).

$A \wedge B$ = logical conjunction. (A and B).

$A \Rightarrow B$ = implication. (A implies B).

$A \equiv B$ = material equivalence (A is materially equivalent to B).

$A \in B$ = set membership. (A is an element of set B).

$A \cap B$ = intersection of sets A and B.

J = Jacobian Determinant.
$\mathbb{R}^p$ = Cartesian space of dimension p.
f = mapping from x-t space to $\rho - U$ space.
f^{-1} = inverse of function f .
g = mapping from $\rho - U$ space to x-t space.

$A(c)$, $M(c)$, $B(c)$ = coefficient matrices defined at point c in domain $\mathscr{D}$ of a function (See Section 3).

$\overline{A}(d)$, $\overline{N}(d)$, $\overline{M}(d)$, $\overline{B}(d)$ = coefficient matrices defined at point d in range R of a function (See Section 3).

I = identity matrix for an $\mathbb{R}^2$ system (See context).

I = impulse function (See context).

ϕ = null matrix for an $\mathbb{R}^2$ system.

m_{ij} = element of matrix M located in the i' th row and the j' th column.

$(m_{ij}) \equiv$ matrix M .

$\overline{m_{ij}}$ = element of matrix $\overline{M}$ located in the i' th row and the j' th column.

$(\overline{m_{ij}}) \equiv$ matrix $\overline{M}$.

b_{ij} = element of matrix B located in the i' th row and the j' th column.

$\overline{b_{ij}}$ = element of matrix $\overline{B}$ located in the i' th row and the j' th column.

b_I = scalar associated with the first of two characteristic directions in x-t space.

b_{II} = scalar associated with the second of two characteristic directions in x-t space.

$\overline{b_I}$ = scalar associated with the first of two characteristic directions in $\rho - U$ space.

$\overline{b_{II}}$ = scalar associated with the second of two characteristic directions in $\rho - U$ space.

$\overline{p}$ = a particular point in range R of a function signifying the intersection of characteristic curves.

Γ_I = characteristic curve of type I in domain $\mathscr{D}$, in x-t space, of function f.

Γ_{II} = characteristic curve of type II in domain $\mathscr{D}$, in x-t space, of function f.

σ_I = curve parameter for a Γ_I characteristic (a length parameter in this context).

σ_{II} = curve parameter for a Γ_{II} characteristic (a length parameter in this context).

$\overline{\Gamma_I}$ = characteristic curve of type I in range R, in $\rho - U$ space, of function f.

$\overline{\Gamma_{II}}$ = characteristic curve of type II in range R, in $\rho - U$ space, of function f.

$\overline{\sigma_I}$ = curve parameter for a $\overline{\Gamma_I}$ characteristic (a length parameter in this context).

$\overline{\sigma_{II}}$ = curve parameter for a $\overline{\Gamma_{II}}$ characteristic (a length parameter in this context).

α_I = linearly independent tangent vector to characteristic curve of type I at a point in domain $\mathscr{D}$, in x-t space, of function f.

α_{II} = linearly independent tangent vector to characteristic curve of type II at a point in domain $\mathscr{D}$, in x-t space, of function f.

$\overline{\alpha_I}$ = linearly independent tangent vector to characteristic curve of type I at a point in range R, in $\rho - U$ space, of function f.

LIST OF SYMBOLS (continued)

$\overline{\alpha_{II}}$ = linearly independent tangent vector to characteristic curve of type II at a point in range R, in $\rho-U$ space, of function f.

$\alpha(\sigma_I)$, $\alpha(\sigma_{II})$ = distinct pairs of functions on distinct characteristic curves of types I and II in domain $\mathscr{D}$, in x-t space, of function f.

$\underline{\alpha}(\overline{\sigma_I})$, $\underline{\alpha}(\overline{\sigma_{II}})$ = distinct pairs of functions on distinct characteristic curves of types I and II in range R, in $\rho-U$ space, of function f.

$\beta(\sigma_I)$, $\beta(\sigma_{II})$ = distinct pairs of functions on distinct characteristic curves of types I and II in domain $\mathscr{D}$, in x-t space, of function f. These functions are Riemann Invariants.

$\underline{\beta}(\overline{\sigma_I})$, $\underline{\beta}(\overline{\sigma_{II}})$ = distinct pairs of functions on distinct characteristic curves of types I and II in range R, in $\rho-U$ space, of function f. These functions are also Riemann Invariants.

$G(\sigma_{II})$, $H(\sigma_{II})$ = distinct functions on characteristic curves of type II in domain $\mathscr{D}$, in x-t space, of function f. The definitions of these functions are used in the special case of a simple wave.

$A(\sigma_I)$, $A(\sigma_{II})$ = distinct pairs of functions on distinct characteristic curves of types I and II in domain $\mathscr{D}$, in x-t space, of function f. The definitions of these functions are specialized for the characteristics in the domain of a simple wave.

$B(\sigma_I)$, $B(\sigma_{II})$ = distinct pairs of functions on distinct characteristic curves of types I and II in domain $\mathscr{D}$, in x-t space, of function f. The definitions of these functions are specialized for the Riemann Invariants associated with a simple wave.

$B_I(\sigma_{II})$ = Riemann Invariant on a type I characteristic.

$B_{II}(\sigma_I)$ = Riemann Invariant on a type II characteristic.

$f: A \rightarrow B$ = function f mapping set A into set B.

$f: A \overset{1-1}{\rightarrow} B$ = function f mapping set A one-to-one into set B.

$f \restriction A$ = function f restricted to set A.

LIST OF SYMBOLS (continued)

A = constant velocity of propagation along a type II characteristic (See context).

$Dmn\, f$ = domain of function f .

ω = scalar ratio following from the vanishing of the Jacobian determinant.

$\mathscr{I}$ = infimum, or lower bound of characteristics in a simple wave.

$\mathscr{s}$ = control volume enclosing $\mathscr{I}$.

T = mapping from x-t space to $T_s - U$ space, where T_s is the local stagnation temperature.

$\{L_\alpha\}_{\alpha=1,2,\ldots}$ = set of lines in domain $\mathscr{D}$ of function f .

θ = origin of coordinates in x-t space.

$C'(y)$ = derivative of differentiable function $C(y)$ with respect to y.

$\Delta\sigma_I$ = interval between two points on a characteristic of type I.

K = ratio of the distance of a given point in interval $\Delta\sigma_I$ from the left-most point in the interval to the length of the interval.

$O(\Delta\sigma_I)^n$ = n' th order error on interval $\Delta\sigma_I$.

$\mathscr{L}$ = control volume having the same cross-section as the void main, extended to the left, into the infinite reservoir.

L = length of control volume $\mathscr{L}$ extending to the left, into the infinite reservoir (See Appendix for this usage).

$L_{-\infty}$ = length of control volume $\mathscr{L}$ extending to $-\infty$.

$\sigma_{0\text{-}0}$ = right-most face of control volume $\mathscr{L}$, which is the surface in which the flow is choked.

σ_+ = remainder of surface of control volume $\mathscr{L}$ when $\sigma_{0\text{-}0}$ is excluded.

σ = entire surface of control volume $\mathscr{L}$ (See Appendix for this usage).

r = ratio of absolute pressure in the quiescent flow region in the reservoir to the absolute static pressure at the plane of choked flow (See Appendix for usage).

β_* = ratio of absolute temperature in the quiescent flow region in the reservoir to the absolute static temperature at the plane of choked flow (See Appendix for usage).

$\beta(z)$ = function of pressure ratio (See Appendix for usage).

ACKNOWLEDGEMENTS

The writer would like to thank Dr. Charles MacArthur, Chief of the Turbine Technologies Branch, for his support, advice, and patience during the approximately one-year period over which this analysis was developed. His reading and comments upon early drafts, and his continued interest in the problem despite many competing day-to-day work requirements, were essential components in the successful completion of this report. Thanks are due Dr. Rolf Sondergaard, whose sharp and accurate understanding of the physics of compressible flow, applied through discussions and reading of drafts of this work, provided many of the insights necessary for its completion. Finally, the writer would like to thank Dr. Paavo Sepri, Visiting Scientist, who generously read and commented upon this work, offering insight from his considerable talent and experience.

1. SUMMARY

Temperature data acquired from turbine stator transient flow tests performed in the Turbine Research Facility, operated by the Propulsion Directorate of the Air Force Research Laboratory, show a large initial thermal impulse appearing in each test. This impulse is clearly not related to the performance characteristics of a particular test article, and it raises a question concerning its interpretation with regard to the test data in which it is imbedded.

The flow physics producing the thermal impulse is assumed to be associated with the sudden expansion of the working fluid from a large reservoir into the domain of the test article, which is initially in vacuo. In August of 1997, a one-dimensional mathematical analysis of the wave phenomena associated with the expansion of a perfect gas from an infinite reservoir into a void main was undertaken in order to understand the origin of this impulse, or spike; and to understand the variables upon which it depends. It was understood at the outset that this model would apply only to the very early development of the flow in the actual main supplying the test article with air from the facility's reservoir.

The gas flow from the infinite reservoir into the main was found to be embodied by a simple, centered, backward-facing expansion wave. The limit of characteristics bounding the wave at its tail was found to be a shock, since the Rankine-Hugoniot Relations are satisfied in a non-trivial way across that bound. The lead characteristic of the expansion wave was found to be permanently located at the exit plane of the reservoir following initiation of the flow process. In the one-dimensional flow model, it is kept in place by an isentropic expansion shock which, in the case of an infinite reservoir, does not contradict the Second Law of Thermodynamics, and thus is physically permissible. This assertion is demonstrated in the appendix of this report. The stationary position of the lead characteristic of the backward-facing expansion wave implies that flow velocity is precisely critical at initial time, and for any time interval following initial time, since the main is infinite in the mathematical model.

The stagnation temperature was found to be a strict function of sound speed throughout the domain of the wave, and was found to increase monotonically from a minimum at the critical point, at the head of the wave, to a maximum at the tail of the wave. The ratio of the maximum to the minimum value was found to be fixed, and to have a value of six for standard air. The strict dependence of stagnation temperature upon sound speed was found to be due to the fact that the Riemann Invariant on each spanning characteristic of the simple wave has precisely the same value. The stagnation temperature is fixed, but has a distinct value on each straight characteristic of the simple wave.

It is concluded, based on the findings of this analysis, that the observed thermal spike occurring in the blow-down process is the result of the heat of the compression to form the shock proceeding into the main when the flow is initiated. The calculated stagnation temperature distribution in the rarefaction wave is the artifact of that process. The theory of characteristics does not yield any information on the flow physics associated with the shock, itself, but it accurately portrays the compression process everywhere in the domain of the flow following the shock.

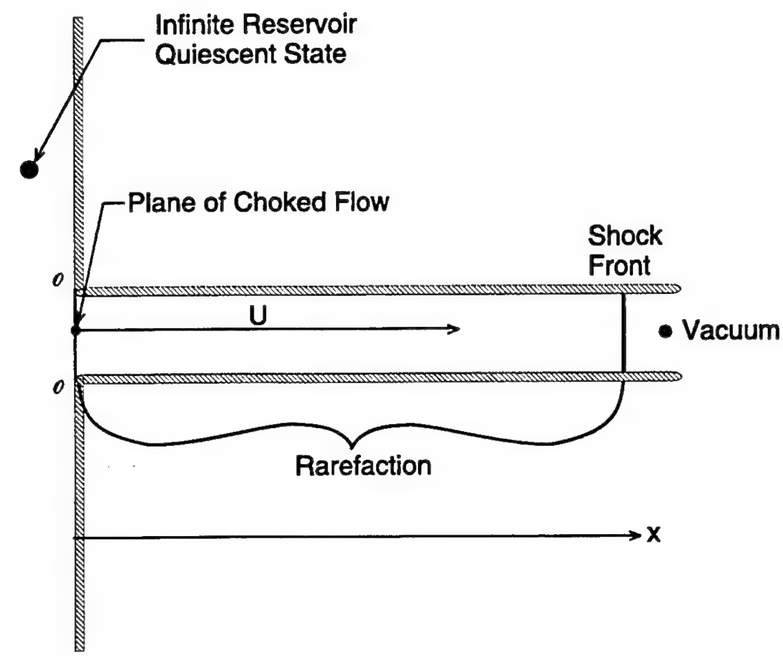
2. INTRODUCTION

The Turbine Research Facility, operated by the Propulsion Directorate of the Air Force Research Laboratory, is a transient blow-down facility which has the purpose of obtaining accurate aerodynamic, thermodynamic, and thermal performance data from large turbine stages without the expenditure of impractical amounts of power. This objective is clearly met by this facility, and it is not the intention of this report to discuss its configuration or operation. This is well documented in open literature. However, an interesting thermodynamic phenomenon, sometimes called the "thermal spike", appears at the initiation of the blow-down process. The purpose of this report is to discuss the mathematical physics of the thermal spike in order to better understand its origin.

The general hypothesis presented in this report is that the thermal spike is produced by a backward-facing rarefaction wave which has, at its tail, the initial shock proceeding into the domain of the test article when the blow-down process is initiated. The mathematical model presented here is that of a one dimensional, centered, backward-facing rarefaction wave created by the expansion of a calorically perfect gas from a reservoir of infinite extent into a void main. The gas flow is assumed to be inviscid.

3. THE FIRST AND SECOND LAWS OF THERMODYNAMICS AND THE DERIVATION OF THE EQUATIONS OF FLUID MOTION

Consider Figure 1. Here is shown an infinite reservoir from which a shock, followed by a rarefaction, issues into a main in vacuo at time 0. The flow is assumed to be one-dimensional, and perpetually choked at the indicated plane 0-0 in Figure 1 after time 0. Since the flow is one-dimensional, there can be no convergence of the streamlines as the flow enters the rarefaction; so the flow cannot accelerate on a gradient as it approaches the plane of choked flow. It is therefore quiescent to the left of plane 0-0, maintaining a time-steady stagnation



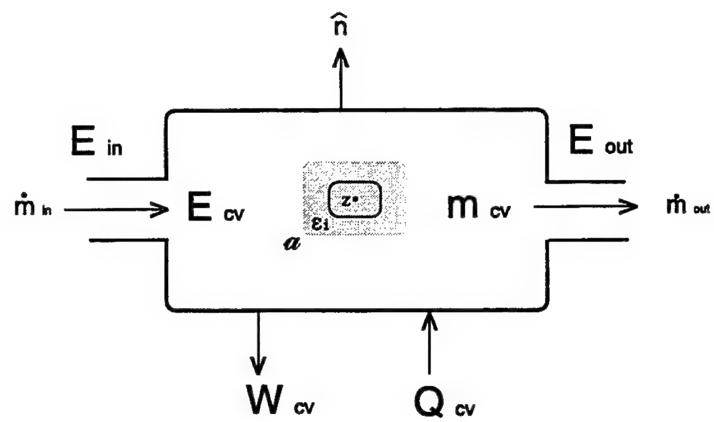
Gas Expansion Into Vacuous Main
Figure 1

thermodynamic state, defined by fixed stagnation temperature and pressure. It turns out that essentially no inconsistency[1] is produced by assuming an abrupt acceleration to critical flow at plane 0-0 in Figure 1, followed by supersonic flow throughout the rarefaction. The thermal spike is evidently produced by the flow work done on the various elements of the expansion wave, also called a rarefaction wave, which propagate at different speeds in the rarefied region.

The analysis of the wave propagation problem requires that we examine the First Law of Thermodynamics in its convective form. We begin by considering control volume V shown in Figure 2. Over any given time interval, $\delta t = t_2 - t_1$, beginning at state 1 and ending at state 2, the First Law of Thermodynamics may be stated according to:

$$E_{in} + Q_{cv} = W_{cv} + (E_2 - E_1)_{cv} + E_{out} \tag{1}$$

where W_{cv} represents the work in all forms done by the system upon its environment. We may assume that our system is adiabatic, and that W_{cv} is strictly flow work. Let $\delta m_{in} = (m_2 - m_1)_{in}$, and $\delta m_{out} = (m_2 - m_1)_{out}$. Clearly,



$$(m_{cv})_2 - (m_{cv})_1 = \delta m_{in} - \delta m_{out}$$

Control Volume V
Figure 2

In this case, over time interval δt, it may be written that:

$$W_{cv} = - p_{in} \cdot v_{in} \cdot \delta m_{in} + p_{out} \cdot v_{out} \cdot \delta m_{out} \tag{2}$$

Note that the specific total energy is defined as:

$$e = u + \frac{(\vec{V} \cdot \vec{V})}{2} \tag{3}$$

[1] Refer to Appendix.

Keeping Equation (3) in mind, we should write:

$$(E_2 - E_1)_{cv} = \delta(m_{cv} \cdot e_{cv}) \tag{4}$$

where m_{cv} and e_{cv} are the mean values in control volume V.

Also:

$$E_{in} = \delta m_{in} \cdot e_{in} \tag{5.i}$$

$$E_{out} = \delta m_{out} \cdot e_{out} \tag{5.ii}$$

In view of the adiabatic hypothesis, Equation (1) implies that the following rate equation exists:

$$\frac{\delta m_{in}}{\delta t} \cdot (e_{in} + p_{in} \cdot v_{in}) = \frac{\delta E_{cv}}{\delta t} + \frac{\delta m_{out}}{\delta t} \cdot (e_{out} + p_{out} \cdot v_{out})$$

Each of the rates in the above equation has a limit as δt becomes infinitesimal, resulting in the expression written next:

$$\frac{\partial E_{cv}}{\partial t} + \frac{\partial m_{out}}{\partial t} \cdot (e_{out} + p_{out} \cdot v_{out}) - \frac{\partial m_{in}}{\partial t} \cdot (e_{in} + p_{in} \cdot v_{in}) = 0 \tag{6}$$

But if σ is the surface area of control volume V, it must be that:

$$\frac{\partial m_{out}}{\partial t} \cdot (e_{out} + p_{out} \cdot v_{out}) - \frac{\partial m_{in}}{\partial t} \cdot (e_{in} + p_{in} \cdot v_{in}) = \iint_\sigma (\rho \vec{V} \cdot \hat{n}) \cdot (e_{surface} + pv)\, d\sigma \tag{7}$$

The role of the flow work upon the rate of change of total energy within V is made apparent by Equation (7).

Consider point z in control volume V of Figure 2. Let us suppose that every neighborhood $\mathcal{a}$ of z, including control volume V, is a regular, simply-connected domain upon which $\vec{V}$ is continuously differentiable, and upon which[1] e, p, and ρ are of class C^1. Let us further assume that z is in an Eulerian reference frame, with the fluid elements flowing past at velocity $\vec{V}$. Let $d\varepsilon$ be an infinitesimal element of $\mathcal{a}$, and pick control volume V itself as an example. Then Gauss' Divergence Theorem implies that for V:

$$\iint_{\sigma}(\rho\vec{V}\cdot\hat{n})\cdot(e_{surface}+pv)\,d\sigma = \iint_{\sigma}(\rho\vec{V}\cdot\hat{n})\cdot e\,d\sigma + \iint_{\sigma}p\cdot(\vec{V}\cdot\hat{n})\,d\sigma =$$
$$\iiint_{V}[\nabla\cdot(\rho\,\vec{V})\cdot e + (\rho\,\vec{V})\cdot\nabla e]\,d\varepsilon + \iiint_{V}[\vec{V}\cdot\nabla p + (\nabla\cdot\vec{V})\cdot p]\,d\varepsilon \tag{8}$$

Equation (8) applies to any neighborhood $\mathcal{a}$ of z.

It is apparent that if one picks another point in V, such as z', then every neighborhood $\mathcal{a}'$ of z' is also a regular, simply-connected domain upon which $\vec{V}$ is continuously differentiable, and upon which e, p, and ρ are of class C^1. Again, we may select control volume V as the example, and let us divide it into a set P of contiguous, non-intersecting sub-volumes ε_i, called a partition, such that each contains a point in V in its interior. It is clear from the properties assigned to V that for a real number $\epsilon > 0$, partition P may be constructed such that the sum shown in Relation (9), taken over a refinement of P, satisfies the indicated condition:

$$\left|\sum_{i\ i=1,\alpha}\frac{\partial(\rho e)}{\partial t}\cdot\varepsilon_i - \frac{\partial E_{cv}}{\partial t}\right| < \epsilon \tag{9}$$

where α is the number of sub-volumes in the refinement. It follows that:

[1] A function on a domain $\mathscr{D}$ in $\mathbb{R}^p$ is of class C^1 if all of its partial derivatives exist and are continuous on $\mathscr{D}$.

$$\frac{\partial E_{cv}}{\partial t} = \iiint_V \frac{\partial(\rho e)}{\partial t}\, d\varepsilon \tag{10}$$

Introducing the substantial derivative, Equation (10) becomes:

$$\frac{\partial E_{cv}}{\partial t} = \iiint_V \rho \cdot \frac{De}{Dt}\, d\varepsilon - \iiint_V \nabla \cdot (\rho\, e\, \vec{V})\, d\varepsilon \tag{11}$$

But we find that Equation (8) may also be written in the form:

$$\begin{aligned} &\iint_\sigma (\rho \vec{V} \cdot \hat{n}) \cdot e\, d\sigma + \iint_\sigma p \cdot (\vec{V} \cdot \hat{n})\, d\sigma = \\ &\iiint_V \nabla \cdot (\rho e\, \vec{V})\, d\varepsilon + \iiint_V \nabla \cdot (p \vec{V})\, d\varepsilon \end{aligned} \tag{12}$$

Hence, Equation (6) requires that:

$$\iiint_V \left[\rho \cdot \frac{De}{Dt} + \nabla \cdot (p\, \vec{V})\right] d\varepsilon = 0 \tag{13}$$

Equation (13) guarantees that at any point z' chosen in V, the following convective[1] form of the energy equation holds:

$$\rho \cdot \frac{De}{Dt} + \nabla \cdot (p\, \vec{V}) = 0 \tag{14}$$

For suppose, at some point z' in control volume V, there exists some number κ such that $|\kappa| > 0$, and such that:

[1] Convective forms are those containing the substantial derivative.

$$\rho \cdot \frac{De}{Dt} + \nabla \cdot (p\vec{V}) = \kappa$$

Then there is a sub-volume, $\varepsilon_i \subseteq V$, and a neighborhood of type $\mathcal{a}^{\prime}$, such that $\varepsilon_i \subseteq \mathcal{a}^{\prime}$;

and which has the property that, for some $|\vartheta| > 0$:

$$\iiint_{\varepsilon_i} \left[\rho \cdot \frac{De}{Dt} + \nabla \cdot (p\vec{V})\right] d\varepsilon = \vartheta$$

Note that under the hypotheses used to formulate Equation (13), control volume V was picked without loss of generality, i.e., it was not assigned any special properties. It follows that, for sub-volume ε_i :

$$\iiint_{\varepsilon_i} \left[\rho \cdot \frac{De}{Dt} + \nabla \cdot (p\vec{V})\right] d\varepsilon \equiv 0$$

Hence, our supposition leads to a contradiction. Equation (14) characterizes the relationship between local flow work done and specific total energy at a point in a flow field for an adiabatic flow process; which is the reason for its derivation in this report.

We reiterate that by convective form of an equation, we mean that it is written in terms of its substantial derivatives. There are four convective equations which apply from the physics principles governing the flow field being considered here. They are listed next.

First Law of Thermodynamics

$$\frac{De}{Dt} + \frac{1}{\rho}\nabla \cdot (p\vec{V}) = 0 \qquad \textbf{(15.i)}$$

Momentum Equations

$$\frac{D\vec{V}}{Dt} + \frac{1}{\rho}\nabla p + \Lambda = 0 \qquad \Lambda \geq 0 \qquad \textbf{(15ii)}$$

Mass Flow Continuity

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V} = 0 \tag{15.iii}$$

Second Law of Thermodynamics

$$T\cdot\frac{Ds}{Dt} = \frac{Du}{Dt} + p\cdot\frac{Dv}{Dt} \tag{15.iiii}$$

Continuing, we note that the above system of equations can be reduced. Note that Equation (15.iiii) can be rewritten in the form:

$$T\cdot\frac{Ds}{Dt} = \frac{Du}{Dt} - \frac{p}{\rho^2}\cdot\frac{D\rho}{Dt} \tag{16}$$

Making use of the mass flow continuity equation, this equation becomes:

$$T\cdot\frac{Ds}{Dt} = \frac{Du}{Dt} + \frac{p}{\rho}\nabla\cdot\vec{V} \tag{17}$$

In view of Equation (3) the substantial derivative of the internal energy may be written as:

$$\frac{Du}{Dt} = \frac{De}{Dt} - 2\left(\frac{\vec{V}}{2}\right)\cdot\frac{D\vec{V}}{Dt} \tag{18}$$

Taking account of Equation (15.ii), the substantial derivative of the internal energy can be written in terms of the momentum constraint:

$$\frac{Du}{Dt} = \frac{De}{Dt} + \vec{V}\cdot\left(\frac{1}{\rho}\nabla p\right) + \vec{V}\cdot\Lambda \tag{19}$$

The result of these operations is that Equation (15.iiii) becomes:

$$T \cdot \frac{Ds}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\nabla \cdot (p\vec{V}) + \vec{V} \cdot \Lambda = \vec{V} \cdot \Lambda \tag{20}$$

Making use of Gauss' Divergence Theorem, Equation (20) implies that for sub-volume, $\varepsilon_i \subseteq V$, the entropy efflux meets the condition that:

$$\iiint_{\varepsilon_i} \left[\frac{\partial(\rho s)}{\partial t} - \frac{\rho \vec{V} \cdot \Lambda}{T} \right] d\varepsilon + \iint_{\sigma_i} s(\rho \vec{V}) \cdot \hat{n} \, d\sigma = 0 \tag{21}$$

It is immediately apparent from the form of Equation (21) that the only lost work is due to the dissipation force per unit volume, Λ. It follows that in case $\Lambda = 0$, i.e., the Euler Equations hold, the lost work function must vanish, and the thermodynamic process must be reversible. If we suppose that this is the case, then our hypothesis that the system is adiabatic implies that:

$$\frac{\partial Q_{\varepsilon_i}}{\partial t} = \iiint_{\varepsilon_i} \left(\rho T \frac{\partial s}{\partial t} \right) d\varepsilon = 0 \tag{22}$$

Hence, if the Euler Equations hold throughout control volume V, and the adiabatic assumption holds[1], then, at every point, p, in V:

$$\frac{\partial s}{\partial t} = 0$$

In addition, Equation (20)[1] is reduced to:

$$\frac{Ds}{Dt} = 0$$

[1] These relations apply in a domain where $\rho T > 0$.

at every point in V. Now recall that:

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s$$

Hence, we must conclude that the following equation holds at every point in V:

$$\vec{V}\cdot\nabla s = 0 \tag{23}$$

Let us add the hypothesis that the Euler Equations of Motion hold for our description of the rarefaction shown in Figure 1. It is now clear that the convective relations which apply from the physics principles governing the flow field are three in number. These governing equations[1] are:

Momentum (Euler Equations)

$$\frac{D\vec{V}}{Dt} + \frac{1}{\rho}\nabla p = 0 \tag{24.i}$$

Mass Flow Continuity

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V} = 0 \tag{24.ii}$$

Entropy Convection

$$\begin{aligned} \frac{Ds}{Dt} &= 0 \\ \frac{\partial s}{\partial t} &= 0 \end{aligned} \tag{24.iii}$$

Finally, we note that for a compressible medium in which pressure is strictly dependent upon density, sound speed is expressible as:

$$C = \sqrt{\frac{dp}{d\rho}} \tag{25}$$

[1] These relations apply in a domain where $\rho T>0$.

4. CHARACTERISTIC THEORY APPLIED TO THE ONE-DIMENSIONAL WAVE

Letting the subscripted variable represent partial differentiation, the inviscid equations of fluid motion described in Section 3, when reduced to one space dimension, appear as follows:

$$\rho_t + U \cdot \rho_x + \rho \cdot U_x = 0 \qquad \textbf{(26.i)}$$

$$U_t + U \cdot U_x + \frac{1}{\rho} \cdot p_x = 0 \qquad \textbf{(26.ii)}$$

$$s_t = 0 \qquad \textbf{(26.iii)}$$

$$U \cdot s_x = 0 \qquad \textbf{(26.iiii)}$$

Clearly, Equations (26.iii) and (26.iiii) imply that the one-dimensional expansion wave described here is isentropic, since $U > 0$ everywhere. When Equation (25) is introduced, the above system reduces to two equations in the following form:

$$\rho_t + U \cdot \rho_x + \rho \cdot U_x = 0 \qquad \textbf{(27.i)}$$

$$\rho \cdot U_t + \rho \cdot U \cdot U_x + C^2 \cdot \rho_x = 0 \qquad \textbf{(27.ii)}$$

We suppose that the partial derivatives, ρ_x, ρ_t, U_x, and U_t all exist, and are continuous throughout domain, $\mathscr{D}$, shown in Figure 3. Then function, f, solving the above equations, is a mapping from domain, $\mathscr{D}$, in the (x, t) plane to range, R, in the (ρ ,U) plane which belongs to class C^1 everywhere on $\mathscr{D}$. It is clear that f is differentiable at every point, such as c, in domain, $\mathscr{D}$. We note that

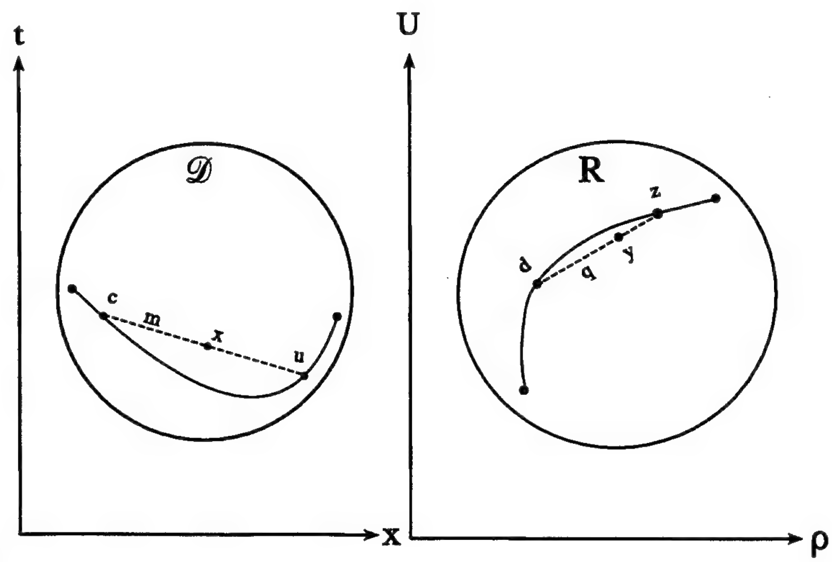


Figure 3

$\mathscr{D}$ and R are each in an $\mathbb{R}^2$ space.

Consider the trajectory passing through point, c, and another point, u, in $\mathscr{D}$, and let m be the distance from c to any point, x, on the straight line connecting c and u. Now, let $\boldsymbol{e}$ be the ordered pair of numbers, (v_1, v_2), such that $m\cdot e = (m\cdot v_1,\ m\cdot v_2)$, and $x = c+m\cdot e$. Continuing, let $\mathrm{D}f(c)$ symbolize the derivative of f at c, and note that $\mathrm{D}f(c)$ is a linear function at point c in $\mathscr{D}$. Let us define the product, $\mathrm{D}f(c)\cdot(m\cdot e)$, as:

$$Df(c)\cdot(m\cdot e) = m \cdot \begin{bmatrix} \rho_x(c) & \rho_t(c) \\ U_x(c) & U_t(c) \end{bmatrix} \cdot \begin{bmatrix} v_1(c) \\ v_2(c) \end{bmatrix} \tag{28}$$

We note that existence of $\mathrm{D}f(c)$ implies that the following inequality must hold:

$$\forall\ \epsilon>0,\ \exists\delta(\epsilon)>0,\ |m\cdot e|<\delta(\epsilon) \Rightarrow |f(c+m\cdot e) - f(c) - Df(c)\cdot(m\cdot e)| \leq \epsilon|m\cdot e| \tag{29}$$

It follows immediately that:

$$\forall\ \epsilon>0,\ \exists\delta(\epsilon)>0,\ |m\cdot e|<\delta(\epsilon) \Rightarrow \left|\frac{1}{m}\cdot\{f(c+m\cdot e) - f(c)\} - Df(c)\cdot(e)\right| \leq \epsilon|e| \tag{30}$$

Inequality (30) proves that $\mathrm{D}f(c)\cdot(e)$ is the directional derivative of f at c in the direction of u. It is not generally the case that the directional derivative of f in the direction of u is tangent to the trajectory at c in $\mathscr{D}$, but as u approaches c on this trajectory, it becomes so. Letting $\boldsymbol{h}$ be the ordered pair of numbers, (w_1, w_2), comprising the components of $\mathrm{D}f(c)\cdot(e)$, we may write:

$$Df(c)\cdot(e) = \begin{bmatrix} w_1(c) \\ w_2(c) \end{bmatrix} = \begin{bmatrix} \rho_x(c) & \rho_t(c) \\ U_x(c) & U_t(c) \end{bmatrix} \cdot \begin{bmatrix} v_1(c) \\ v_2(c) \end{bmatrix} \tag{31}$$

The Jacobian Determinant for the above transformation should be written as seen next.

$$J = \begin{vmatrix} \rho_x(c) & \rho_t(c) \\ U_x(c) & U_t(c) \end{vmatrix} \tag{32}$$

Let us suppose that $J \neq 0$. In this case, the following equations may be written:

$$\begin{bmatrix} v_1(c) \\ v_2(c) \end{bmatrix} = \frac{1}{J} \cdot \begin{bmatrix} U_t(c) & -\rho_t(c) \\ -U_x(c) & \rho_x(c) \end{bmatrix} \cdot \begin{bmatrix} w_1(c) \\ w_2(c) \end{bmatrix} \tag{33}$$

It is clear that if, and only if, $J \neq 0$, Equations (31) and (33) require that the linear function, $\mathrm{D}f(c)$, be a one-one correspondence mapping the Cartesian space, $\mathbb{R}^2$, containing $\mathcal{D}$, onto the Cartesian space, $\mathbb{R}^2$, containing R.

Let us introduce that the following theorem:

Inversion Theorem. Suppose function, f is in class C^1 on a neighborhood of $c \in \mathbb{R}^p$ mapping $\mathbb{R}^p$ to $\mathbb{R}^p$. Let us further suppose that $\mathrm{D}f(c)$ is a one-one map of $\mathbb{R}^p$ onto $\mathbb{R}^p$. Then there exists V(U) such that U is a neighborhood of c, and V is a neighborhood of $f(c)$, and f is a one-one mapping of U onto V, and f has a continuous inverse function, g, defined on set V to set U. Moreover, g is in class C^1 on V, and if $y \in V$ and $x = g(y) \in U$, then the linear function, $\mathrm{D}g(y)$, is the inverse of linear function, $\mathrm{D}f(x)$.

Note that the hypothesis that $\mathrm{D}f(c)$ is a one-one map of $\mathbb{R}^p$ onto $\mathbb{R}^p$ is satisfied for $p = 2$ in our system. The Inversion Theorem guarantees the existence of a function g mapping domain R to domain $\mathcal{D}$ such that g is continuous on R, and such that $g = f^{-1}$ on R. We conclude, in addition, that if $d \in R$, and $c = g(d) \in D$, then $\mathrm{D}g(d)$, is the inverse of $\mathrm{D}f(c)$. Consider $d \in R$, and let the trajectory in R, portrayed in Figure 3, pass through points d and z in R, where $d = f(c)$ and $z = f(u)$. By the Inversion Theorem, we are assured that $c = g(d)$ and $u = g(z)$. Let h be a particular ordered pair, $(w_{1(c)}, w_{2(c)})$, and note that for point y on the straight line connecting points d and z in R, we may select a positive, real number, q, such that $q \cdot h = (q \cdot w_{1(c)}, q \cdot w_{2(c)})$, and $y = d + q \cdot h$. It is thus apparent that q is the distance from d to y, and that $h(gd)) \equiv h(c)$. Using a construction similar to that applied to $\mathrm{D}f(c)$ in domain $\mathcal{D}$, the directional derivative of g at d in the direction of z in R may be shown to exist, and be written as:

$$Dg(d) \cdot (h) = \begin{bmatrix} v_1(g(d)) \\ v_2(g(d)) \end{bmatrix} = \begin{bmatrix} v_1(c) \\ v_2(c) \end{bmatrix} = \frac{1}{J} \cdot \begin{bmatrix} U_t(g(d)) & -\rho_t((g(d)) \\ -U_x((g(d)) & \rho_x((g(d)) \end{bmatrix} \cdot \begin{bmatrix} w_1((g(d)) \\ w_2((g(d)) \end{bmatrix} \tag{34}$$

As was the case for $\mathrm{D}f(c)\cdot(e)$, $\mathrm{D}g(d)\cdot(h)$ becomes tangent to the trajectory shown in R as point z approaches point d on that trajectory.

One may deduce from Figure 3 and Equation (34) that when z is located such that the line connecting d and z is parallel to the axis of abscissas, then $w_2 = 0$, and when the line connecting d and z is parallel to the axis of ordinates, $w_1 = 0$. In case $w_2 = 0$, Equation (34) implies that:

$$x_\rho(d) = \frac{U_t(c)}{J} \tag{35.i}$$

$$t_\rho(d) = -\frac{U_x(c)}{J} \tag{35.ii}$$

In case $w_1 = 0$, Equation (34) implies that:

$$x_U(d) = -\frac{\rho_t(c)}{J} \tag{36.i}$$

$$t_U(d) = \frac{\rho_x(c)}{J} \tag{36.ii}$$

Equations (35.i), (35.ii), (36.i), and (36.ii) yield the relations between the components of $\mathrm{D}f(c)$, and $\mathrm{D}g(d)$ whenever $J \neq 0$. It immediately follows that function g solves the pair of first-order partial differential equations, seen next, everywhere on R if, and only if, $J \neq 0$.

$$x_U - U\cdot t_U + \rho\cdot t_\rho = 0 \tag{37.i}$$

$$\rho\cdot x_\rho - \rho\cdot U\cdot t_\rho + C^2\cdot t_U = 0 \tag{37.ii}$$

Equations (37.i) and (37.ii) are linear, as opposed to Equations (27.i) and (27.ii), which are not. Initially, this fact would seem to be important; but it turns out to be simply a curiosity.

In order to effectively solve Equations (27.i), (27.ii); and (37.i), (37.ii), we must discuss the properties of these equation systems. Consider the derivatives of f at point c in

$\mathscr{D}$, and g at point d in R, and let us write Equations (27.i), (27.ii), and (37.i), (37.ii) in matrix form. We arrange these equations in a style to accommodate this, as shown below. The form for Equations (27.i), (27.ii) is:

$$\begin{aligned} &\{U\cdot\rho_x+\rho\cdot U_x\} + \{\rho_t + 0\} = 0 \\ &\left\{\frac{C^2}{\rho}\cdot\rho_x+U\cdot U_x\right\} + \{0 + U_t\} = 0 \end{aligned} \tag{38}$$

The corresponding forms for Equations (37.i). (37.ii) are:

$$\begin{aligned} &\left\{\frac{\rho}{C^2}\cdot x_\rho - \frac{\rho}{C^2}\cdot U\cdot t_\rho\right\} + \{0 + t_U\} = 0 \\ &\{0 + t_\rho\} + \left\{\frac{1}{\rho}\cdot x_U - \frac{1}{\rho}\cdot U\cdot t_U\right\} = 0 \end{aligned} \tag{39}$$

Note that the partial derivatives of f at point c in $\mathscr{D}$ may be represented as the column vectors listed following:

$$\frac{\partial f}{\partial x}(c) = \begin{bmatrix} \rho_x(c) \\ U_x(c) \end{bmatrix} \tag{40.i}$$

and,

$$\frac{\partial f}{\partial t}(c) = \begin{bmatrix} \rho_t(c) \\ U_t(c) \end{bmatrix} \tag{40.ii}$$

The corresponding partial derivatives of g at point d in R are also represented as column vectors, written as:

$$\frac{\partial g}{\partial \rho}(d) = \begin{bmatrix} x_\rho(d) \\ t_\rho(d) \end{bmatrix} \tag{41.i}$$

and,

$$\frac{\partial g}{\partial U}(d) = \begin{bmatrix} x_U(d) \\ t_U(d) \end{bmatrix} \tag{41.ii}$$

Consider point c in domain $\mathscr{D}$, and point d in range R corresponding to c under f. Equations (38) and (39) motivate the definition of the following matrices at these points:

$$A(c) = \begin{bmatrix} U(c) & \rho(c) \\ \left(\dfrac{C^2}{\rho}(c)\right) & U(c) \end{bmatrix} \tag{42}$$

and,

$$\overline{A}(d) = \begin{bmatrix} \left(\dfrac{\rho}{C^2}\right)(d) & -\left(\dfrac{\rho}{C^2}\cdot U\right)(d) \\ 0 & 1 \end{bmatrix} \tag{43}$$

We also require:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{44}$$

and,

$$\overline{N}(d) = \begin{bmatrix} 0 & 1 \\ \left(\dfrac{1}{\rho}\right)(d) & -\left(\dfrac{1}{\rho}\cdot U\right)(d) \end{bmatrix} \tag{45}$$

Finally, we define matrix ϕ as:

$$\phi = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{46}$$

Then Equations (27.i), and (27.ii) may be written as the following matrix equation at point c in domain $\mathscr{D}$:

$$A \cdot \frac{\partial f}{\partial x} + I \cdot \frac{\partial f}{\partial t} = \phi \tag{47}$$

The corresponding matrix equation for Equations (37.i), and (37.ii) at point d in R are:

$$\overline{A} \cdot \frac{\partial g}{\partial \rho} + \overline{N} \cdot \frac{\partial g}{\partial U} = \phi \tag{48}$$

Note that:

$$Df(c) \cdot (e) = \frac{\partial f}{\partial x}(c) \cdot v_1(c) + \frac{\partial f}{\partial t}(c) \cdot v_2(c) \tag{49}$$

and:

$$Dg(d) \cdot (h) = \frac{\partial g}{\partial \rho}(d) \cdot w_1(g(d)) + \frac{\partial g}{\partial U}(d) \cdot w_2(g(d)) \tag{50}$$

where v_1, v_2 are real numbers, and are the components of e in $\mathscr{D}$. The real numbers, w_1, w_2 are the corresponding components of h in R. Letting M be a matrix of elements, (m_{ij}), multiplying the terms of Equation (47), and $\overline{M}$ be a matrix of elements, $(\overline{m_{ij}})$, multiplying the terms of Equation (48), we note that Equation (47) is equivalent to:

$$v_2 \cdot M \cdot A \cdot \frac{\partial f}{\partial x} \cdot v_1 + v_1 \cdot M \cdot \frac{\partial f}{\partial t} \cdot v_2 = \phi \tag{51}$$

and that Equation (48) is equivalent to:

$$w_2 \cdot \overline{M} \cdot \overline{A} \cdot \frac{\partial g}{\partial \rho} \cdot w_1 + w_1 \cdot \overline{M} \cdot \overline{N} \cdot \frac{\partial g}{\partial U} \cdot w_2 = \phi \tag{52}$$

In the event that M can be found such that its elements, (m_{ij}), are not necessarily 0, and such that M is not singular, and $v_2 \cdot M \cdot A = v_1 \cdot M = B$, then the directional derivative of f in the direction of u from c, solving Equations (27.i), and (27.ii), on $\mathscr{D}$ satisfies:

$$B \cdot Df(c) \cdot (e) = \phi \tag{53}$$

If Equation (53) holds, and M is not singular, then the direction of u from c is *characteristic* of Equation System (27.i), and (27.ii), and the differential equation solved by f on a trajectory in that direction is ordinary. Since M is a matrix of rank 2, there are *two* linearly independent directions in $\mathscr{D}$ in which u can be located with respect to c which are characteristic.

In similar fashion, if one can find a matrix, $\overline{M}$, such that $w_2 \cdot \overline{M} \cdot \overline{A} = w_1 \cdot \overline{M} \cdot \overline{N} = \overline{B}$, and $\overline{M}$ is not singular, then the directional derivative of g in the direction of z from d, solving Equations (37.i), and (37.ii), on R satisfies:

$$\overline{B} \cdot Dg(d) \cdot (h) = \phi \tag{54}$$

Matrix, $\overline{M}$, is also of rank 2; hence there are *two* linearly independent directions in which z can be located with respect to d in R which are characteristic, as well.

The condition, $v_2 \cdot M \cdot A = v_1 \cdot M = B$, requires that:

$$\begin{bmatrix} \left(m_{11} \cdot v_2 \cdot U + m_{12} \cdot v_2 \cdot \frac{C^2}{\rho}\right) & (m_{11} \cdot v_2 \cdot \rho + m_{12} \cdot v_2 \cdot U) \\ \left(m_{21} \cdot v_2 \cdot U + m_{22} \cdot v_2 \cdot \frac{C^2}{\rho}\right) & (m_{21} \cdot v_2 \cdot \rho + m_{22} \cdot v_2 \cdot U) \end{bmatrix} = \begin{bmatrix} m_{11} \cdot v_1 & m_{12} \cdot v_1 \\ m_{21} \cdot v_1 & m_{22} \cdot v_1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{55}$$

Corresponding to Equation (54), the condition, $w_2 \cdot \overline{M} \cdot \overline{A} = w_1 \cdot \overline{M} \cdot \overline{N} = \overline{B}$, requires that:

$$\begin{bmatrix} \left(w_2 \cdot \frac{\rho}{C^2} \cdot \overline{m_{11}}\right) & \left(-w_2 \cdot \frac{\rho}{C^2} \cdot U \cdot \overline{m_{11}} + w_2 \cdot \overline{m_{12}}\right) \\ \left(w_2 \cdot \frac{\rho}{C^2} \cdot \overline{m_{21}}\right) & \left(-w_2 \cdot \frac{\rho}{C^2} \cdot U \cdot \overline{m_{21}} + w_2 \cdot \overline{m_{22}}\right) \end{bmatrix} =$$

$$\begin{bmatrix} \left(w_1 \cdot \frac{1}{\rho} \cdot \overline{m_{12}}\right) & \left(w_1 \cdot \overline{m_{11}} - w_1 \cdot \frac{1}{\rho} \cdot U \cdot \overline{m_{12}}\right) \\ \left(w_1 \cdot \frac{1}{\rho} \cdot \overline{m_{22}}\right) & \left(w_1 \cdot \overline{m_{21}} - w_1 \cdot \frac{1}{\rho} \cdot U \cdot \overline{m_{22}}\right) \end{bmatrix} = \begin{bmatrix} \overline{b_{11}} & \overline{b_{12}} \\ \overline{b_{21}} & \overline{b_{22}} \end{bmatrix} \tag{56}$$

Equation (55), in fact, yields the constraints upon (m_{ij}) if M is not to be singular. First, Equation (55) requires that these constraints satisfy Equation pairs (57) and (58), seen next. The conditions imposed for m_{11} and m_{12} are:

$$\begin{aligned} m_{11}(v_2 \cdot U - v_1) + m_{12} \cdot v_2 \cdot \frac{C^2}{\rho} &= 0 \\ m_{11} \cdot v_2 \cdot \rho + m_{12} \cdot (v_2 \cdot U - v_1) &= 0 \end{aligned} \tag{57}$$

The corresponding conditions imposed for m_{21} and m_{22} are:

$$m_{21}(v_2 \cdot U - v_1) + m_{22} \cdot v_2 \cdot \frac{C^2}{\rho} = 0$$

$$m_{21} \cdot v_2 \cdot \rho + m_{22} \cdot (v_2 \cdot U - v_1) = 0 \qquad \textbf{(58)}$$

Equation (56) yields the constraints upon $(\overline{m_{ij}})$ if $\overline{M}$ is not to be singular. This equation requires that the elements of $(\overline{m_{ij}})$ satisfy Equation Pairs (59) and (60) following. The conditions on $\overline{m_{11}}$ and $\overline{m_{12}}$ are:

$$\overline{m_{11}} \cdot w_2 \cdot \frac{\rho}{C^2} - \overline{m_{12}} \cdot w_1 \cdot \frac{1}{\rho} = 0$$

$$\overline{m_{11}} \cdot \left(w_2 \cdot \frac{\rho}{C^2} \cdot U + w_1 \right) - \overline{m_{12}} \cdot \left(w_2 + w_1 \cdot \frac{1}{\rho} \cdot U \right) = 0 \qquad \textbf{(59)}$$

The corresponding conditions on $\overline{m_{21}}$ and $\overline{m_{22}}$ are:

$$\overline{m_{21}} \cdot w_2 \cdot \frac{\rho}{C^2} - \overline{m_{22}} \cdot w_1 \cdot \frac{1}{\rho} = 0$$

$$\overline{m_{21}} \cdot \left(w_2 \cdot \frac{\rho}{C^2} \cdot U + w_1 \right) - \overline{m_{22}} \cdot \left(w_2 + w_1 \cdot \frac{1}{\rho} \cdot U \right) = 0 \qquad \textbf{(60)}$$

It is clear that M is not constrained to be singular if and, only if, the following determinant vanishes:

$$\begin{vmatrix} (v_2 \cdot U - v_1) & v_2 \cdot \dfrac{C^2}{\rho} \\ v_2 \cdot \rho & (\cdot v_2 \cdot U - v_1) \end{vmatrix} = 0 \tag{61}$$

On the other hand, matrix $\overline{M}$ is not constrained to be singular if and, only if:

$$\begin{vmatrix} w_2 \cdot \dfrac{\rho}{C^2} & -w_1 \cdot \dfrac{1}{\rho} \\ \left(w_2 \cdot \dfrac{\rho}{C^2} \cdot U + w_1 \right) & -\left(w_2 + w_1 \cdot \dfrac{1}{\rho} \cdot U \right) \end{vmatrix} = 0 \tag{62}$$

Note that Equation (61) requires that:

$$v_2 \cdot (U \pm C) - v_1 = 0 \tag{63}$$

Suppose $v_2 \cdot (U + C) - v_1 = 0$. In this case, $(v_2 \cdot U - v_1) = -v_2 \cdot C$, and Equations (57) require:

$$m_{12} = m_{11} \cdot \frac{\rho}{C} \tag{64}$$

Matrix M will be singular unless we incorporate the other form of Equation (63) in Equations (58). The other form is: $v_2 \cdot (U - C) - v_1 = 0$, which means that $(v_2 \cdot U - v_1) = v_2 \cdot C$;

whence:

$$m_{21} = -m_{22} \cdot \frac{C}{\rho} \tag{65}$$

Let $b_I = m_{11} \cdot v_1$ and $b_{II} = m_{22} \cdot v_1$. Matrix B may now be expressed as:

$$B = \begin{bmatrix} b_I & b_I \cdot \frac{\rho}{C} \\ -b_{II} \cdot \frac{C}{\rho} & b_{II} \end{bmatrix} \tag{66}$$

It is apparent from the form of matrix B that its elements are associated with the characteristic directions in $\mathscr{D}$ as illustrated Table 1.

Table 1

b_I:	$v_2 \cdot (U + C) - v_1 = 0$
b_{II}:	$v_2 \cdot (U - C) - v_1 = 0$

In each of the characteristic directions associated with b_I and b_{II} respectively, Matrix Equation (53) yields a linearly independent form depending upon $\mathrm{D} f(c) \cdot (e)$, written as follows:

$$\begin{aligned} b_I \cdot \left(\rho_x + \frac{\rho}{C} \cdot U_x \right) \cdot v_1 + b_I \cdot \left(\rho_t + \frac{\rho}{C} \cdot U_t \right) \cdot v_2 &= 0 \\ b_{II} \cdot \left(- \frac{C}{\rho} \cdot \rho_x + U_x \right) \cdot v_1 + b_{II} \cdot \left(- \frac{C}{\rho} \cdot \rho_t + U_t \right) \cdot v_2 &= 0 \end{aligned} \tag{67}$$

On the other hand, Equation (62) requires that:

$$(C\cdot w_1 - \rho\cdot w_2)\cdot(C\cdot w_1 + \rho\cdot w_2) = 0 \tag{68}$$

Equation (68) implies either $(C\cdot w_1 - \rho\cdot w_2) = 0$ or $(C\cdot w_1 + \rho\cdot w_2) = 0$. In the case of matrix $\overline{M}$, if the first condition holds, then:

$$\overline{m_{12}} = \frac{\rho}{C}\cdot\overline{m_{11}} \tag{70}$$

As in the case of matrix M, $\overline{M}$ will be singular unless the second condition is applied to define the elements of its second row. Consequently,

$$\overline{m_{21}} = -\frac{C}{\rho}\cdot\overline{m_{22}} \tag{71}$$

Using Equations (70) and (71), we may write Equation (56) as:

$$w_2\cdot\overline{M}\cdot\overline{A} = w_1\cdot\overline{M}\cdot\overline{N} = \begin{bmatrix} \left(\overline{m_{11}}\cdot\frac{\rho}{C^2}\right)\cdot w_2 & \left(-\overline{m_{11}}\cdot\frac{\rho}{C^2}\cdot U + \overline{m_{11}}\cdot\frac{\rho}{C}\right)\cdot w_2 \\ \left(-\overline{m_{22}}\cdot\frac{1}{C}\right)\cdot w_2 & \left(\overline{m_{22}}\cdot\frac{U}{C} + \overline{m_{22}}\right)\cdot w_2 \end{bmatrix} \tag{72}$$

Defining $\overline{b_I} = \overline{m_{22}}\cdot\frac{1}{C}\cdot w_2$ and $\overline{b_{II}} = \overline{m_{11}}\cdot\frac{\rho}{C}\cdot w_2$, matrix $\overline{B}$ may be written as:

$$\overline{B} = \begin{bmatrix} \overline{b}_{II}\cdot\frac{1}{C} & -\overline{b}_{II}\cdot\left(\frac{U}{C} - 1\right) \\ -\overline{b}_{I} & \overline{b}_{I}\cdot(U + C) \end{bmatrix} \tag{73}$$

Hence, the characteristic directions for $g = f^{-1}$, solving Equations (37.i) and (37.ii), may be associated with $\overline{b}_I$ and $\overline{b}_{II}$ as follows in Table 2:

Table 2

$\overline{b}_I$:	$(C\cdot w_1 + \rho\cdot w_2) = 0$
$\overline{b}_{II}$:	$(C\cdot w_1 - \rho\cdot w_2) = 0$

In this case, Matrix Equation (54) yields a linearly independent form in each of the characteristic directions, depending upon $\mathrm{D}g(d)\cdot(h)$, which may be written as:

$$\begin{aligned} &\overline{b}_I\cdot[-x_\rho + (U+C)\cdot t_\rho]\cdot w_1 + \overline{b}_I\cdot[-x_U + (U+C)\cdot t_U]\cdot w_2 = 0 \\ &\overline{b}_{II}\cdot[-x_\rho + (U-C)\cdot t_\rho]\cdot w_1 + \overline{b}_{II}\cdot[-x_U + (U-C)\cdot t_U]\cdot w_2 = 0 \end{aligned} \tag{74}$$

Now consider a particular point, p, in $\mathscr{D}$ and let $\overline{p}$ be its image under f in R. Without loss of generality, we may select the direction in $\mathscr{D}$ associated with b_I. Since f belongs to class C^1, and p is not special, one may apply Inequality (30) to $\mathrm{D}f(p)\cdot(e)$ solving Equation (53) to construct a unique curve, Γ_I, passing through point p in the selected direction. The condition that f belongs to class C^1 implies that Inequality (30) may be used to construct Γ_I as the limit of a Cauchy sequence of continuous curves passing through point p. The limit is the unique curve, Γ_I, passing through point p, which has a

continuously turning tangent in $\mathscr{D}$. Curve Γ_I is a characteristic curve of Equation System (27.i) and (27.ii). It is apparent that Equation (53) admits the construction of precisely one other curve, Γ_{II}, passing through p, which is also a characteristic curve of Equation System (27.i) and (27.ii). Let us associate Γ_I with b_I, and Γ_{II} with b_{II}, in matrix B in the subsequent discussion.

Characteristic curves, Γ_I and Γ_{II}, have images, $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$, in R under f, which intersect at $\bar{p}$. Both of these pairs of curves are illustrated in Figure 4. Here, α_I and α_{II} are linearly independent tangent vectors associated respectively with Γ_I and Γ_{II}. Clearly, curves $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$ are characteristic curves of Equation System (37.i) and (37.ii) provided that $J \neq 0$. In this case, $\overline{\alpha_I}$ and $\overline{\alpha_{II}}$ are linearly independent tangent vectors associated respectively with $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$. Obviously, Γ_I and Γ_{II} are the images of $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$ in $\mathscr{D}$ under g, when g exists.

Note that the existence of $g = f^{-1}$, i.e. $J \neq 0$, is not essential to the construction of Γ_I and Γ_{II} in $\mathscr{D}$. This development leads to a class of special wave functions which solve Equation (53), but do not require the condition, $J \neq 0$. The expansion of a perfect gas from an infinite reservoir into a void main is described by one of this class of special wave functions, which is described in Section 5.

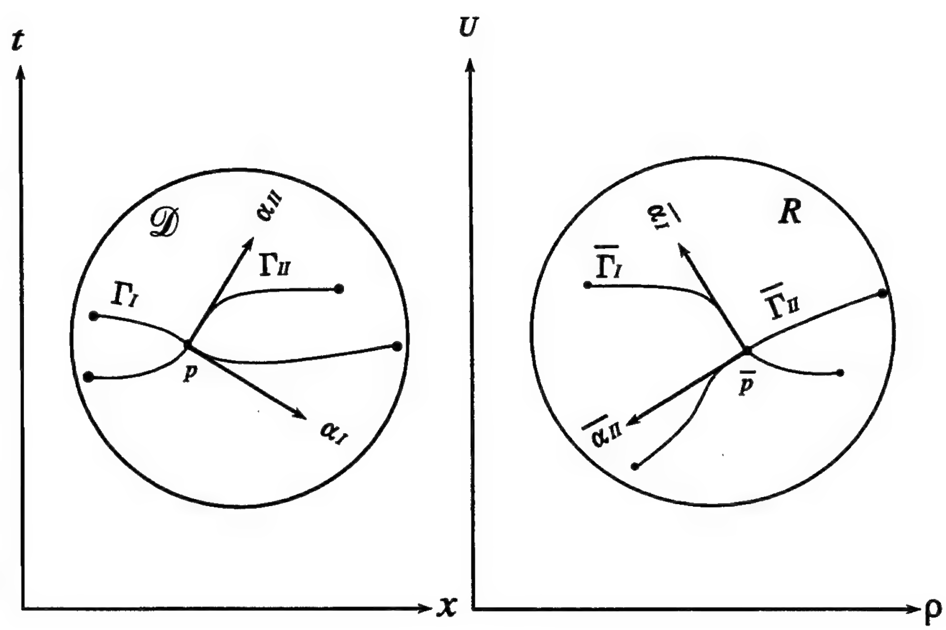


Figure 4

Continuing with the condition that $J \neq 0$, let σ_I and σ_{II} be distance parameters associated with Γ_I and Γ_{II}, respectively; and let $\overline{\sigma_I}$ and $\overline{\sigma_{II}}$ be distance parameters associated with $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$. In domain $\mathscr{D}$, we may make the following table of associations between the characteristic directions and derivatives with respect to the distance parameters:

Table of Associated Characteristic Directions and Total Derivatives

Γ_I: $\qquad v_1 = \dfrac{dx}{d\sigma_I}$

Γ_I: $\qquad v_2 = \dfrac{dt}{d\sigma_I}$

and

Γ_{II}: $\qquad v_1 = \dfrac{dx}{d\sigma_{II}}$

Γ_{II}: $\qquad v_2 = \dfrac{dt}{d\sigma_{II}}$

Corresponding Associations in Region R

$\overline{\Gamma}_I$: $\qquad w_1 = \dfrac{d\rho}{d\overline{\sigma}_I}$

$\overline{\Gamma}_I$: $\qquad w_2 = \dfrac{dU}{d\overline{\sigma}_I}$

and

$\overline{\Gamma}_{II}$: $\qquad w_1 = \dfrac{d\rho}{d\overline{\sigma}_{II}}$

$\overline{\Gamma}_{II}$: $\qquad w_2 = \dfrac{dU}{d\overline{\sigma}_{II}}$

Now at point p in domain $\mathscr{D}$, Table 1, found on page 22, demonstrates that the domain points of function f solving Equations (27.i) and (27.ii) satisfy the ordinary first-order differential equations shown in Table 3, in the indicated characteristic directions.

Table 3

Γ_I:
$$\frac{dx}{d\sigma_I} - (U+C)\cdot\frac{dt}{d\sigma_I} = 0$$

Γ_{II}:
$$\frac{dx}{d\sigma_{II}} - (U-C)\cdot\frac{dt}{d\sigma_{II}} = 0$$

Equations (67) yield first-order, ordinary differential equations for the corresponding range points of f in the directions of Γ_I and Γ_{II}. These Equations are illustrated in Table 4.

Table 4

Γ_I:
$$\frac{d\rho}{d\sigma_I} + \frac{\rho}{C}\cdot\frac{dU}{d\sigma_I} = 0$$

Γ_{II}:
$$\frac{d\rho}{d\sigma_{II}} - \frac{\rho}{C}\cdot\frac{dU}{d\sigma_{II}} = 0$$

In case $J \neq 0$, function g exists such that $g = f^{-1}$. Set R, in the (ρ ,U) plane, is then in the domain of g, and g solves Equations (37.i) and (37.ii) on R. At point $\overline{p}$ in set R, Table 2, found on page 24, demonstrates that the domain points of function g solving Equations (37.i) and (37.ii) satisfy the ordinary, first-order differential equations shown in Table 5, below, in the characteristic directions, $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$, in R.

Table 5

$\overline{\Gamma_I}$:
$$\frac{d\rho}{d\overline{\sigma_I}} + \frac{\rho}{C}\cdot\frac{dU}{d\overline{\sigma_I}} = 0$$

$\overline{\Gamma_{II}}$:
$$\frac{d\rho}{d\overline{\sigma_{II}}} - \frac{\rho}{C}\cdot\frac{dU}{d\overline{\sigma_{II}}} = 0$$

Equations (74) yield first-order, ordinary differential equations for the corresponding range points of g in the directions of $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$. These Equations appear in Table 6 following.

Table 6

$$\overline{\Gamma}_I: \qquad \frac{dx}{d\overline{\sigma}_I} - (U + C)\cdot\frac{dt}{d\overline{\sigma}_I} = 0$$

$$\overline{\Gamma}_{II}: \qquad \frac{dx}{d\overline{\sigma}_{II}} - (U - C)\cdot\frac{dt}{d\overline{\sigma}_{II}} = 0$$

The Second Law of Thermodynamics and the Perfect Equation of State for a calorically perfect gas may be embodied in Tables 4 and 5 to produce particularly simple first-order differential equations. In each characteristic direction in $\mathcal{D}$, at a point, we write the following associated forms, noting that[1] R is the gas constant for the Perfect Equation of State in Tables 7 and 8:

Table 7

$$\Gamma_I: \qquad p(\sigma_I) = \rho(\sigma_I)\cdot R\cdot T(\sigma_I) \qquad T\cdot\frac{ds}{d\sigma_I} = \frac{\gamma}{\gamma-1}\cdot R\cdot\frac{dT}{d\sigma_I} - \frac{1}{\rho}\cdot\frac{dp}{d\sigma_I}$$

$$\Gamma_{II}: \qquad p(\sigma_{II}) = \rho(\sigma_{II})\cdot R\cdot T(\sigma_{II}) \qquad T\cdot\frac{ds}{d\sigma_{II}} = \frac{\gamma}{\gamma-1}\cdot R\cdot\frac{dT}{d\sigma_{II}} - \frac{1}{\rho}\cdot\frac{dp}{d\sigma_{II}}$$

In each characteristic direction in set[1] R, the domain of function g , at a point, we write the following corresponding forms:

Table 8

$$\overline{\Gamma}_I: \qquad p(\overline{\sigma}_I) = \rho(\overline{\sigma}_I)\cdot R\cdot T(\overline{\sigma}_I) \qquad T\cdot\frac{ds}{d\overline{\sigma}_I} = \frac{\gamma}{\gamma-1}\cdot R\cdot\frac{dT}{d\overline{\sigma}_I} - \frac{1}{\rho}\cdot\frac{dp}{d\overline{\sigma}_I}$$

$$\overline{\Gamma}_{II}: \qquad p(\overline{\sigma}_{II}) = \rho(\overline{\sigma}_{II})\cdot R\cdot T(\overline{\sigma}_{II}) \qquad T\cdot\frac{ds}{d\overline{\sigma}_{II}} = \frac{\gamma}{\gamma-1}\cdot R\cdot\frac{dT}{d\overline{\sigma}_{II}} - \frac{1}{\rho}\cdot\frac{dp}{d\overline{\sigma}_{II}}$$

[1] R refers to the range of f except when used in the context of the Perfect Equation of State.

From our hypothesis that the Euler Equations of Motion hold, it has been shown that the wave is isentropic. Hence, the thermodynamic relations of Table 7 result in the following ordinary, first-order differential equations for the range points of f , holding in each of the characteristic directions in $\mathscr{D}$:

Table 9

Γ_I:	$\frac{d\rho}{\sigma_I} = \frac{2}{\gamma-1}\cdot\frac{\rho}{C}\cdot\frac{dC}{d\sigma_I}$	$\frac{d\rho}{d\sigma_I} + \frac{\rho}{C}\cdot\frac{dU}{d\sigma_I} = 0$
Γ_{II}:	$\frac{d\rho}{\sigma_{II}} = \frac{2}{\gamma-1}\cdot\frac{\rho}{C}\cdot\frac{dC}{d\sigma_{II}}$	$\frac{d\rho}{d\sigma_{II}} - \frac{\rho}{C}\cdot\frac{dU}{d\sigma_{II}} = 0$

For completeness, Table 10 illustrates the ordinary, first-order differential equations in each characteristic direction in set R, the domain of inverse function, g .

Table 10

$\overline{\Gamma}_I$:	$\frac{d\rho}{\overline{\sigma}_I} = \frac{2}{\gamma-1}\cdot\frac{\rho}{C}\cdot\frac{dC}{d\overline{\sigma}_I}$	$\frac{d\rho}{d\overline{\sigma}_I} + \frac{\rho}{C}\cdot\frac{dU}{d\overline{\sigma}_I} = 0$
$\overline{\Gamma}_{II}$:	$\frac{d\rho}{\overline{\sigma}_{II}} = \frac{2}{\gamma-1}\cdot\frac{\rho}{C}\cdot\frac{dC}{d\overline{\sigma}_{II}}$	$\frac{d\rho}{d\overline{\sigma}_{II}} - \frac{\rho}{C}\cdot\frac{dU}{d\overline{\sigma}_{II}} = 0$

Through Tables 1 - 10, one can see that function f , solving Equations (27.i) and (27.ii), is transformed through a one-to-one correspondence to characterize the solutions to the following set of ordinary, first-order differential equations which appear in Table 11:

Table 11

Characteristic	Domain $\mathscr{D}$	Range $\underline{\mathbf{R}}$[1]
Γ_I:	$\dfrac{dx}{d\sigma_I} - (U+C)\cdot\dfrac{dt}{d\sigma_I} = 0$	$\dfrac{2}{\gamma-1}\cdot\dfrac{dC}{d\sigma_I} + \dfrac{dU}{d\sigma_I} = 0$
Γ_{II}:	$\dfrac{dx}{d\sigma_{II}} - (U-C)\cdot\dfrac{dt}{d\sigma_{II}} = 0$	$\dfrac{2}{\gamma-1}\cdot\dfrac{dC}{d\sigma_{II}} - \dfrac{dU}{d\sigma_{II}} = 0$

The corresponding ordinary, first-order differential equations applying to inverse function g are illustrated in Table 12; again, for the sake of completeness.

Table 12

Characteristic	Range $\mathscr{D}$	Domain $\underline{\mathbf{R}}$[1]
$\overline{\Gamma}_I$:	$\dfrac{dx}{d\overline{\sigma}_I} - (U+C)\cdot\dfrac{dt}{d\overline{\sigma}_I} = 0$	$\dfrac{2}{\gamma-1}\cdot\dfrac{dC}{d\overline{\sigma}_I} + \dfrac{dU}{d\overline{\sigma}_I} = 0$
$\overline{\Gamma}_{II}$:	$\dfrac{dx}{d\overline{\sigma}_{II}} - (U-C)\cdot\dfrac{dt}{d\overline{\sigma}_{II}} = 0$	$\dfrac{2}{\gamma-1}\cdot\dfrac{dC}{d\overline{\sigma}_{II}} - \dfrac{dU}{d\overline{\sigma}_{II}} = 0$

Letting [2] $\alpha(\sigma_I)$, $\beta(\sigma_I)$; and $\alpha(\sigma_{II})$, $\beta(\sigma_{II})$, be strict functions of the distance, or curve parameters of characteristic curves, Γ_I , and Γ_{II} , and requiring that each of these distinct function pairs be defined at each point on each of the characteristic

[1] $\underline{\mathbf{R}}$ is range of f transformed according to the relations of Table 7.

[2] $\alpha(\sigma_I)$, $\alpha(\sigma_{II})$; and $\beta(\sigma_I)$, $\beta(\sigma_{II})$ are, respectively, distinct pairs of functions.

curves, Γ_I , and Γ_{II} , indefinite integrals may be written on these curves in the form seen in Table 13:

Table 13

Characteristic	Domain $\mathscr{D}$	Range $\underline{\mathbf{R}}$[1]
Γ_I:	$x - \int (U+C)\cdot dt = \alpha(\sigma_{II})$	$\frac{2}{\gamma-1}\cdot C + U = \beta(\sigma_{II})$
Γ_{II}:	$x - \int (U-C)\cdot dt = \alpha(\sigma_I)$	$\frac{2}{\gamma-1}\cdot C - U = \beta(\sigma_I)$

Table 13 may be thought of as the restriction of f to Γ_I, written as $f \upharpoonright \Gamma_I$, in the first row, and as the restriction of f to Γ_{II}, written as $f \upharpoonright \Gamma_{II}$, in the second row.

The corresponding result for $g = f^{-1}$ is shown in Table 14, where[2] $\underline{\alpha}(\overline{\sigma_I})$, $\underline{\beta}(\overline{\sigma_I})$; and $\underline{\alpha}(\overline{\sigma_{II}})$, $\underline{\beta}(\overline{\sigma_{II}})$, are the corresponding strict functions of the distance, or curve parameters of characteristic curves, $\overline{\Gamma_I}$, and $\overline{\Gamma_{II}}$:

Table 14

Characteristic	Range $\mathscr{D}$	Domain $\underline{\mathbf{R}}$[1]
$\overline{\Gamma_I}$:	$x - \int (U+C)\cdot dt = \underline{\alpha}(\overline{\sigma_{II}})$	$\frac{2}{\gamma-1}\cdot C + U = \underline{\beta}(\overline{\sigma_{II}})$
$\overline{\Gamma_{II}}$:	$x - \int (U-C)\cdot dt = \underline{\alpha}(\overline{\sigma_I})$	$\frac{2}{\gamma-1}\cdot C - U = \underline{\beta}(\overline{\sigma_I})$

[1] $\underline{\mathbf{R}}$ is the range of f transformed according to the relations of Table 7.

[2] $\underline{\alpha}(\overline{\sigma_I})$, $\underline{\alpha}(\overline{\sigma_{II}})$; and $\underline{\beta}(\overline{\sigma_I})$, $\underline{\beta}(\overline{\sigma_{II}})$ are, respectively, distinct pairs of functions

Table 14 may be thought of as the restriction of g to $\overline{\Gamma_I}$, written as $g \upharpoonright \overline{\Gamma_I}$, in the first row, and as the restriction of g to $\overline{\Gamma_{II}}$, written as $g \upharpoonright \overline{\Gamma_{II}}$, in the second row.

Noting that for each point p in $\mathscr{D}$, the domain of f solving Equations (27.i) and (27.ii), has two characteristic curves, Γ_I and Γ_{II}, passing through it in linearly independent directions, and a single pair of distance or curve parameters, σ_I and σ_{II}, associated with it, one may create a mapping from the (x, t) pair to the (σ_I, σ_{II}) pair associated with each point in $\mathscr{D}$. Let $\mathscr{C}$ be the set of (σ_I, σ_{II}) pairs under this mapping. This is an injective mapping, written symbolically as :

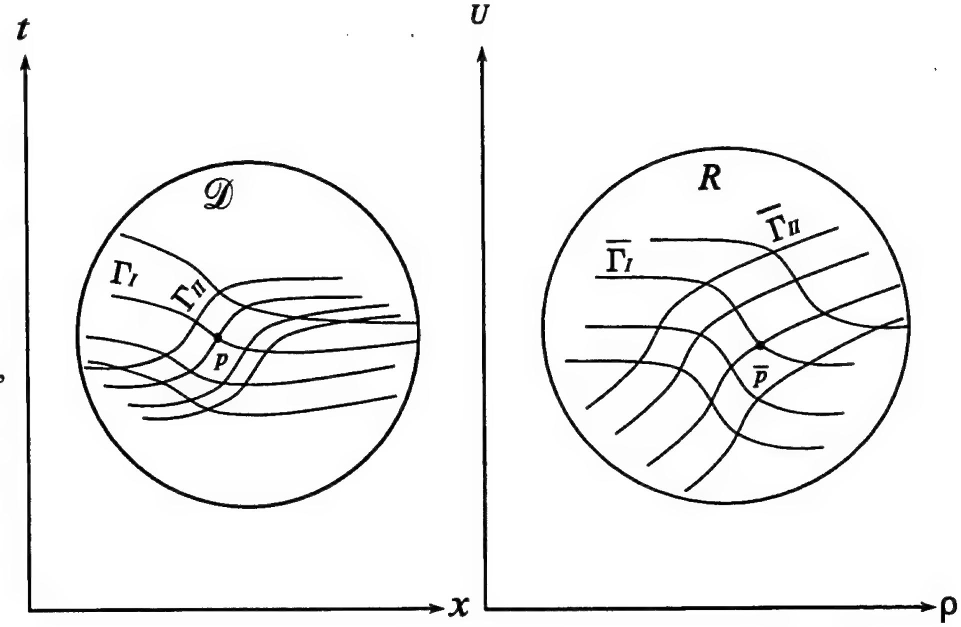


Figure 5

$$\mathscr{M} : \mathscr{D} \overset{1-1}{\rightarrow} \mathscr{C} \qquad (75)$$

Referring to Figure 5, suppose we choose to move along the characteristic curve Γ_I passing through p . In this case, curve parameter σ_I varies with position on Γ_I, but σ_{II} remains unaffected, since the other characteristic direction is linearly independent.

Let $\mathscr{N} = \mathscr{M} \upharpoonright \Gamma_I$, and let $\mathscr{C}_{\Gamma_I}$ be the range of $\mathscr{N}$. Then $Dmn\ \mathscr{N} \cap \mathscr{D} = \Gamma_I$ and $Rng\ \mathscr{N} \cap \mathscr{C} = \mathscr{C}_{\Gamma_I}$. Using set builder notation, $\mathscr{N}$ may be described as:

$$\mathscr{N} = \{(n, m) \mid [n = (x, t) \wedge n \in \Gamma_I \wedge m = (\sigma_I, \sigma_{II}) \wedge m \in \mathscr{C}_{\Gamma_I}] \wedge [n_1 = n_2 \Rightarrow m_1 = m_2]\}$$

Clearly, $\mathscr{N}$ must have the property that $\forall\ n_1, n_2,\ \ n_1, n_2 \in \Gamma_I \Rightarrow (\sigma_{II})_1 = (\sigma_{II})_2$ if Γ_{II} is not to be simply a fixed point. Evidently, $\alpha(\sigma_{II})$ and $\beta(\sigma_{II})$ have fixed values on characteristic Γ_I. We could, as well, have chosen characteristic curve Γ_{II} to construct $\mathscr{N}$; hence, we conclude that $\alpha(\sigma_I)$ and $\beta(\sigma_I)$ have fixed values on characteristic Γ_{II}, also.

Turning again to Figure 5, let us consider point $\bar{p}$ in R, the image of point p in $\mathscr{D}$ under f. Now R is the domain of g, and so $\bar{p}$ has two characteristic curves, $\overline{\Gamma_I}$ and $\overline{\Gamma_{II}}$, passing through it in linearly independent directions, and a distinct pair of distance or curve parameters, $\overline{\sigma_I}$ and $\overline{\sigma_{II}}$, associated with it, just as was the case with p in $\mathscr{D}$. Choosing $\overline{\Gamma_I}$, and letting $\overline{\mathscr{C}}$ be the set of pairs, $(\overline{\sigma_I}, \overline{\sigma_{II}})$, associated with the pairs, (ρ, U), along $\overline{\Gamma_I}$, we may write the following injective mapping on R:

$$\overline{\mathscr{M}}: R \overset{1-1}{\rightarrow} \overline{\mathscr{C}} \tag{76}$$

Proceeding in the same manner with function g as we did with function f, let $\overline{\mathscr{N}} = \overline{\mathscr{M}} \upharpoonright \overline{\Gamma_I}$, and let $\overline{\mathscr{C}}_{\overline{\Gamma_I}}$ be the range of $\overline{\mathscr{N}}$. Then $Dmn\,\overline{\mathscr{N}} \cap R = \overline{\Gamma_I}$ and $Rng\,\overline{\mathscr{N}} \cap \overline{\mathscr{C}} = \overline{\mathscr{C}}_{\overline{\Gamma_I}}$. Using set builder notation, $\overline{\mathscr{N}}$ may be described as:

$$\overline{\mathscr{N}} = \{ (\bar{n}, \bar{m}) \mid [\bar{n} = (\rho, U) \wedge \bar{n} \in \overline{\Gamma_I} \wedge \bar{m} = (\overline{\sigma_I}, \overline{\sigma_{II}}) \wedge \bar{m} \in \overline{\mathscr{C}}_{\overline{\Gamma_I}}] \wedge [\overline{n_1} = \overline{n_2} \Rightarrow \overline{m_1} = \overline{m_2}]\}$$

We find that $\overline{\mathscr{N}}$ has the property that $\forall\ \overline{n_1}, \overline{n_2},\ \ \overline{n_1}, \overline{n_2} \in \overline{\Gamma_I} \Rightarrow (\overline{\sigma_{II}})_1 = (\overline{\sigma_{II}})_2$ if $\overline{\Gamma_{II}}$ is not to be a fixed point. This shows that $\underline{\alpha}(\overline{\sigma_{II}})$ and $\underline{\beta}(\overline{\sigma_{II}})$ have fixed values on characteristic $\overline{\Gamma_I}$. Obviously, we have, without loss of generality, shown that $\underline{\alpha}(\overline{\sigma_I})$ and $\underline{\beta}(\overline{\sigma_I})$ have fixed values on characteristic $\overline{\Gamma_{II}}$, in addition. We note here that $\beta(\sigma_I)$ and $\beta(\sigma_{II})$ are sometimes called the Riemann Invariants.

Continuing, we shall refer to Γ_I and $\overline{\Gamma_I}$ as characteristics of Type I in $\mathscr{D}$; and we shall refer to Γ_{II} and $\overline{\Gamma_{II}}$ as characteristics of Type II in R. The pairs of Type I and II characteristic curves form an untangled net covering domain $\mathscr{D}$ of f, as shown in Figure 5. The pairs of Type I and II characteristic curves form an untangled net covering domain R of $g = f^{-1}$, when it exists. This corresponding net is also shown in Figure 5.

5. SIMPLE, ONE-DIMENSIONAL WAVES

Our characterization of simple waves begins with the following definition.

Definition. Suppose the there exists a domain $\mathscr{D}$ in the (x, t) plane for which f solving Equations (27.i) and (27.ii) belongs to class C^1. [1] Let us suppose that all of the characteristics of a single type covering $\mathscr{D}$, e.g. Type I, are straight lines. Then if R is the plane of (ρ , U) pairs, we say that this function, symbolized by:

$$f : \mathscr{D} \rightarrow R$$

is a simple wave.

Theorem. A function f which belongs to class C^1, and which solves Equations (27.i) and (27.ii) on domain $\mathscr{D}$ in the (x, t) plane, is a simple wave on $\mathscr{D}$ if, and only if, Determinant J, defined by Equation (32), vanishes at every point in $\mathscr{D}$.

Proof. Consider the directional derivative of f at point p in $\mathscr{D}$, and recall its form from Equation (31) in Section 4. We write:

$$Df(p)\cdot(e) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \rho_x & \rho_t \\ U_x & U_t \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{77}$$

Also recall that the determinant of $Df(c)\cdot(e)$ is:

$$J = \begin{vmatrix} \rho_x & \rho_t \\ U_x & U_t \end{vmatrix} \tag{78}$$

[1] A function on a domain $\mathscr{D}$ in $\mathbb{R}^p$ is of class C^1 if all of its partial derivatives exist and are continuous on $\mathscr{D}$.

Now, by Cramer's Rules, the following Equations must hold:

$$J \cdot v_1 = \begin{vmatrix} w_1 & \rho_t \\ w_2 & U_t \end{vmatrix} \tag{79.i}$$

$$J \cdot v_2 = \begin{vmatrix} \rho_x & w_1 \\ U_x & w_2 \end{vmatrix} \tag{79.ii}$$

Suppose $J = 0$; but let us require that an ordered pair, (v_1, v_2) , exists at every point in $\mathscr{D}$. In this case, the following identity holds, provided that ω exists :

$$\frac{U_t}{\rho_t} = \frac{U_x}{\rho_x} = \frac{w_2}{w_1} = \omega \tag{80}$$

Now ω exists everywhere in $\mathscr{D}$ if, and only if, $w_1 \neq 0$ at every point in $\mathscr{D}$; which will be the case if U depends upon ρ everywhere in $\mathscr{D}$. But we are assured by Equation (24.iii) that this condition holds . Also, since Equations (27.i) and (27.ii) hold everywhere in $\mathscr{D}$, the condition that $J = 0$, when applied to them, guarantees that:

$$\omega = \pm \frac{C}{\rho} \tag{81}$$

Without loss of generality, we may set $\omega = + \dfrac{C}{\rho}$ at point p in $\mathscr{D}$. Continuing, the condition that $J = 0$, when applied to Equations (79.i) and (79.ii), requires that:

$$w_1 \cdot U_x - w_2 \cdot \rho_x = 0 \tag{82.i}$$

$$w_1 \cdot U_t - w_2 \cdot \rho_t = 0 \tag{82.ii}$$

We reiterate that the isentropic condition, guaranteed by Equation (24.iii), implies that ω exists everywhere in $\mathscr{D}$. Hence, Equations (82.i) and (82.ii) reduce to the single condition:

$$w_1 - \frac{\rho}{C} \cdot w_2 = 0 \tag{83}$$

Let σ be a general distance parameter in the direction specified by e in $\mathrm{D}f(c)\cdot(e)$, and let us associate the following derivatives with (v_1, v_2) and (w_1, w_2) at a point in domain $\mathscr{D}$:

$$v_1 = \frac{dx}{d\sigma} \qquad w_1 = \frac{d\rho}{d\sigma}$$

$$v_2 = \frac{dt}{d\sigma} \qquad w_2 = \frac{dU}{d\sigma}$$

Then $J = 0$ implies that Equation (83) has the following invariant form, for any direction specified by e in $\mathrm{D}f(c)\cdot(e)$, at point p in $\mathscr{D}$:

$$\frac{d\rho}{d\sigma} - \frac{\rho}{C} \cdot \frac{dU}{d\sigma} = 0 \tag{84}$$

We could, as well, have made the choice[1], $\omega = -\dfrac{C}{\rho}$.

Recall from Section 4 that the existence of $g = f^{-1}$, i.e. $J \neq 0$, is not essential to the construction of Γ_I and Γ_{II} in $\mathscr{D}$. Hence, we may align $\mathrm{D}f(c)\cdot(e)$ at point p in $\mathscr{D}$ with the characteristic solutions of Equation (53), which can be constructed even when $J = 0$. When we do this, the following associations appear in the characteristic directions, as illustrated in Table 15 :

[1] The form of Equation (84) would be $\frac{d\rho}{d\sigma} + \frac{\rho}{C} \cdot \frac{dU}{d\sigma} = 0$ in this case.

Table 15[1]
Characteristic Associations When $J = 0$

Γ_I:

$$\frac{dx}{d\sigma_I} - (U+C)\cdot\frac{dt}{d\sigma_I} = 0$$

$$\frac{d\rho}{d\sigma_I} - \frac{\rho}{C}\cdot\frac{dU}{d\sigma_I} = 0 \quad *$$

$$\frac{d\rho}{d\sigma_I} + \frac{\rho}{C}\cdot\frac{dU}{d\sigma_I} = 0$$

Γ_{II}:

$$\frac{dx}{d\sigma_{II}} - (U-C)\cdot\frac{dt}{d\sigma_{II}} = 0$$

$$\frac{d\rho}{d\sigma_{II}} - \frac{\rho}{C}\cdot\frac{dU}{d\sigma_{II}} = 0 \quad *$$

$$\frac{d\rho}{d\sigma_{II}} - \frac{\rho}{C}\cdot\frac{dU}{d\sigma_{II}} = 0$$

Both ordinary, first-order differential equations for the range points of f on Γ_I apply, so we conclude that on this Type I characteristic, the range points of f satisfy the conditions shown below:

$$\frac{d\rho}{d\sigma_I} = 0 \qquad \rho = G(\sigma_{II})$$

$$\frac{dU}{d\sigma_I} = 0 \qquad U = H(\sigma_{II})$$

where $G(\sigma_{II})$ and $H(\sigma_{II})$ are strict functions of the curve parameter for a characteristic of Type II intersecting Γ_I .

[1] The $*$ appearing in Table 15 indicates the form of Equation (84) when D $f(c)\cdot(e)$ is aligned with the indicated characteristic, and $J = 0$.

Since the characteristic directions at point p are linearly independent, curve parameter σ_{II} is fixed at every point on Γ_I. Since point p was chosen arbitrarily, it follows that when $J = 0$, $G(\sigma_{II})$ and $H(\sigma_{II})$ have fixed values all along Γ_I passing through point p. In view of the condition that the wave is isentropic, we refer to Table 7 and find, consequently, that $\dfrac{dT}{d\sigma_I} = 0$. It follows that $C = \sqrt{\gamma \cdot R \cdot T}$ is necessarily a strict function of σ_{II} on Γ_I. *Hence, Table 15 shows that characteristic* Γ_I *is a straight line.*

Letting $A(\sigma_{II})$ and $B(\sigma_{II})$ be two more strict functions of a curve parameter for a characteristic of Type II intersecting Γ_I, the equation for Γ_I, and the equation for the Riemann Invariant on Γ_I appear on the first line of Table 16. The corresponding forms for the Type I characteristic appear on the second line:

Table 16

Characteristic	**Domain** $\mathscr{D}$	**Range** $\underline{\mathbf{R}}$
Γ_I:	$x - (U + C) \cdot t = A(\sigma_{II})$	$\dfrac{2}{\gamma - 1} \cdot C + U = B(\sigma_{II})$
Γ_{II}:	$x - \int (U - C) \cdot dt = A(\sigma_I)$	$\dfrac{2}{\gamma - 1} \cdot C - U = B(\sigma_I)$

Clearly, $A(\sigma_I)$ and $B(\sigma_I)$; and $A(\sigma_{II})$ and $B(\sigma_{II})$ have fixed values, as well, on the indicated characteristics[1], in their associated rows, in Table 16.

On the other hand, suppose Γ_I is a straight line in $\mathscr{D}$. It can be shown, as a result, that $\dfrac{d\rho}{d\sigma_I} = 0$ and $\dfrac{dU}{d\sigma_I} = 0$ everywhere on Γ_I. Consider a particular point p on Γ_I, and note that the following matrix equation must hold:

[1] $A(\sigma_I)$, $A(\sigma_{II})$; and $B(\sigma_I)$, $B(\sigma_{II})$ are, respectively, distinct pairs of functions.

$$\begin{bmatrix} U_t(p) & -\rho_t(p) \\ -U_x(p) & \rho_x(p) \end{bmatrix} \cdot \begin{bmatrix} w_1(p) \\ w_2(p) \end{bmatrix} = J \cdot \begin{bmatrix} v_1(p) \\ v_2(p) \end{bmatrix} \tag{85}$$

where the pairs , (v_1, v_2) and (w_1, w_2) , are defined on page 12. By hypothesis, Γ_I passes through p in $\mathscr{D}$; thus one of the pair, (v_1, v_2) , must be different from zero. If we require $J \neq 0$, then Equation (85) implies that $w_1 = \dfrac{d\rho}{d\sigma_I}$ and $w_2 = \dfrac{dU}{d\sigma_I}$; but either $|w_1| > 0$ or $|w_1| > 0$, in this case, contradicting our supposition. Hence $J = 0$, and our theorem is proved. ■

Each of the characteristics of Type I form a straight line for our choice of ω in Equation (81). A sketch of the characteristic net for such a wave is shown in Figure 6. The range of f on $\mathscr{D}$ is the single curve in the (ρ , U) plane, depicted as R in Figure 6. Every point on the Γ_I illustrated, such as points p_1, p_2, p_3, and p_4, maps to the single point, $\bar{p}$, on curve R, for the example shown here.

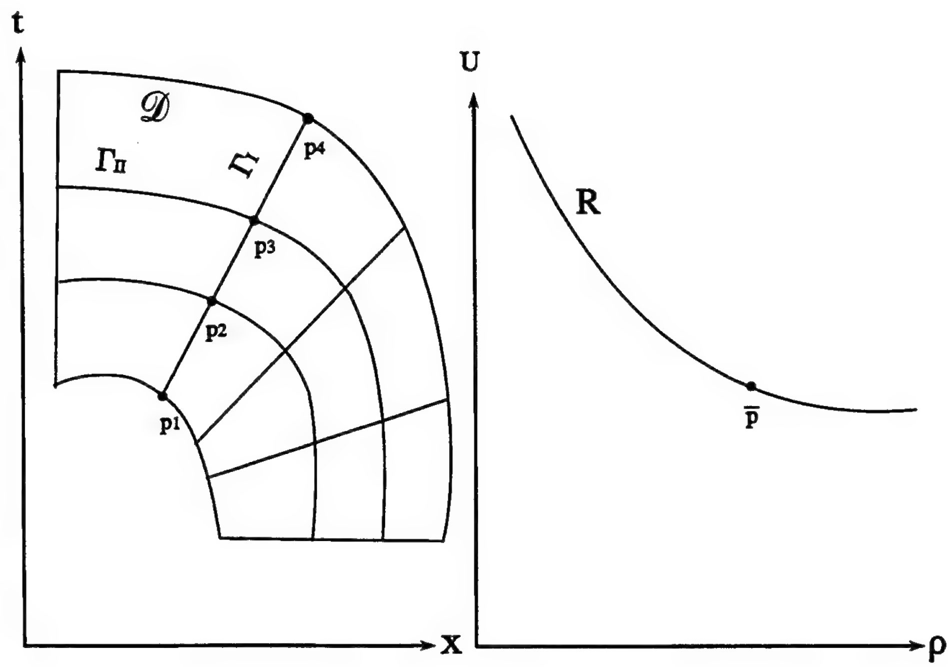


Figure 6

A simple wave whose straight-line characteristics in domain $\mathscr{D}$ are of Type I is a forward-facing wave. If we had chosen $\omega = -\dfrac{C}{\rho}$ at point p in $\mathscr{D}$, the straight-line characteristics would have been of Type II, and the simple wave would have been backward-facing. If all of the straight-line characteristics intersect at a point on the bound of $\mathscr{D}$, the wave is centered.

The head of the wave of interest in this analysis is located at plane 0-0 in Figure 1. It is of characteristic Type II. The flow enters the wave here at critical velocity and proceeds

into the rarefied region at supersonic speed. It is clear that the thermodynamic state on the head characteristic is constant, and that characteristic is, therefore straight. Obviously, every characteristic of Type I in the domain $\mathscr{D}$ of this wave has an intersection with the head characteristic at plane 0-0, and each of these intersections is at the same thermodynamic state. It is easy to show that $J = 0$ at every point p in $\mathscr{D}$, as a consequence. Hence, the subject wave is a *simple, backward-facing, centered expansion wave* - sometimes known as a centered rarefaction wave.

There is a shock at the tail of the backward-facing expansion wave which is not in the domain of the wave, and to which characteristic theory does not apply. We need to establish that this shock exists. Since every characteristic of Type II is straight in domain $\mathscr{D}$, one can show that the infimum $\mathscr{I}$, or greatest lower bound of characteristics , is also a straight line, i.e., it has a constant velocity of propagation. Consider Figure 7 and let a fixed rectangular control volume $\mathscr{A}$ be such that face L of $\mathscr{A}$ is to left of $\mathscr{I}$, and such that face R of $\mathscr{A}$ is to its right. Then control volume $\mathscr{A}$ must proceed at constant velocity, V_{cv} , which is the velocity of propagation of $\mathscr{I}$.

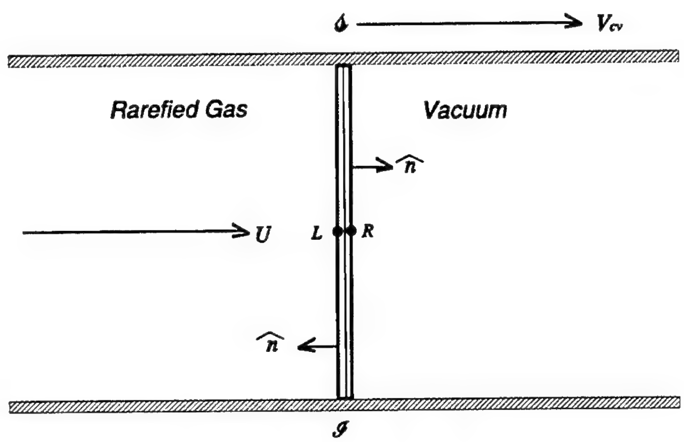


Figure 7

The right face of control volume $\mathscr{A}$ is in vacuo. This being the case, it must be that $p = 0$ at R in Figure 7, where p symbolizes static pressure in this portion of the discussion. Clearly, L is interior to domain $\mathscr{D}$; hence $p > 0$ at face L.

Recall that the flow in domain $\mathscr{D}$ of the expansion wave is isentropic. The fact that the static pressure, p , at a point c in the interior of $\mathscr{D}$ is such that $p(c) > 0$ implies that both $\rho(c) > 0$ and $u(c) > 0$. If the gas is calorically perfect, the thermodynamic properties at point c are related to those at face L by the following two equations :

$$\frac{u_L}{u_{(c)}} = \left(\frac{p_L}{p_{(c)}}\right)^{\frac{\gamma-1}{\gamma}} \tag{86}$$

and

$$\frac{\rho_L}{\rho_{(c)}} = \left(\frac{u_L}{u_{(c)}} \right)^{\frac{1}{\gamma-1}} \tag{87}$$

Continuing, let us establish the following definitions with reference to $\mathscr{s}$, where subscript L refers to the variable value at face L in Figure 7:

Definition	**Face L**	**Face R**
Total Energy	$e_L = u_L + \frac{(U_L - V_{cv})^2}{2}$	$e_R = 0$
Total Enthalpy	$h_L = e_L + \frac{p_L}{\rho_L}$	$h_R = 0$
Impulse	$I_L = -\,p_L - \rho_L \cdot (U_L - V_{cv})^2$	$I_R = 0$
Mass Flux	$m_L = -\,\rho_L \cdot (U_L - V_{cv})$	$m_R = 0$

Clearly, every nested sequence of control volumes $\{\mathscr{s}\}_{i=1,\infty}$, by definition of $\mathscr{s}$, contains $\mathscr{I}$, and has as its limit, $\mathscr{I}$ with faces L and R on its left and right, respectively. Let us refer to this limit as $\mathscr{s}_M$. We now observe that if the gas is calorically perfect, the specific internal energy, and the sound speed C, are related according to Equations (88) and (89), shown next:

$$\frac{p_L}{\rho_L} = (\gamma - 1) \cdot u_L \tag{88}$$

$$C = \sqrt{\frac{\gamma p_L}{\rho_L}} \tag{89}$$

Now in the case of our backward-facing expansion wave, one can show that Riemann Invariant $B(\sigma_{II})$, from Table 16, is fixed everywhere in $\mathscr{D}$. Furthermore, for the limiting $\mathscr{s}$ containing $\mathscr{I}$, i.e., $\mathscr{s}_M$, Equation (86) implies that $u_L = 0$. It follows from Equations (88) and (89) that $U_L = B(\sigma_{II})$ for the limiting case. This velocity is the upper limit, or escape velocity for fluid elements in $\mathscr{D}$. It is clear enough, then, that $V_{cv} = B(\sigma_{II})$ as well. It now follows that with respect to $\mathscr{s}_M$:

$$e_L = e_R = 0 \tag{90.i}$$

$$h_L = h_R = 0 \tag{90.ii}$$

$$I_L = I_R = 0 \tag{90.iii}$$

$$m_L = m_R = 0 \tag{90.iiii}$$

Equations (90.ii), (90.iii), and (90.iiii) are, respectively, conservation of energy, momentum, and mass flow in the reference frame of $\mathscr{I}$. They constitute the Rankine-Hugoniot conditions, and are satisfied with respect to the limit control volume, $\mathscr{s}_M$. Note that they are trivially satisfied under the condition of no mass flux at either face of the wave, as would be the case if $\mathscr{I}$ were a contact surface. This condition would normally require that $U_L = U_R$, making the convective gas velocity continuous at the location of $\mathscr{I}$. However, $\rho_L = \rho_R = 0$ for $\mathscr{s}_M$ containing $\mathscr{I}$, so $m_L = m_R = 0$ even when $U_L \neq U_R$. Since $U_L = B(\sigma_{II})$ at face L, and $U_R = 0$ at face R, it is clear that $\mathscr{I} \subseteq \mathscr{s}_M$ is the site of a velocity discontinuity - an abrupt velocity jump from $U = 0$ to $U = B(\sigma_{II})$ propagating at escape velocity for the expansion wave on domain $\mathscr{D}$. Hence $\mathscr{I}$ is a shock with its front facing the vacuum, and its back facing the expansion wave.

6. CALCULATION OF THE ONE-DIMENSIONAL, BACKWARD-FACING EXPANSION WAVE

Consider Figure 1 from Section 3. We are assuming that the flow is one-dimensional everywhere, and that the reservoir is infinite in extent. Hence, when the rarefaction is initiated at time zero, there can be no influence on the thermodynamic state of the reservoir. There can

be no local flow effects in the gas to the left of plane 0-0 without obviating the one-dimensional flow field hypothesis. This is to say that the reservoir has sufficient depth to maintain a fixed stagnation pressure and temperature everywhere to the left of plane 0-0. Consequently, the reservoir is able to maintain a constant, perpetually choked condition at plane 0-0 after the initiation of the rarefaction. Letting T_0 be the stagnation temperature at plane 0-0, and p_0 be the stagnation pressure there, these thermodynamic quantities are fixed at plane 0-0 for all time following the initiation of the rarefaction. The gas velocity at plane 0-0 is fixed after time zero, as a result, maintaining a value expressed as:

$$U_{cr} = \sqrt{\frac{2\gamma R T_0}{(\gamma+1)}} \tag{91}$$

The head of the wave is adjoined to plane 0-0, which is maintained at a constant thermodynamic state. Hence, $J = 0$ everywhere in domain $\mathscr{D}$ of the wave, with the consequence that it is a *simple wave*. Furthermore, we deduce that the lead characteristic, which is at the head of the wave, must permanently coincide with plane 0-0 after time zero, since the wave cannot penetrate the reservoir and disturb its thermodynamic state. This means that Type II characteristics must be straight, making the wave backward-facing. This means also that there exist curve parameters σ_I and σ_{II} for characteristic families Γ_I and Γ_{II} ; and functions $A_I(\sigma_{II})$ and $A_{II}(\sigma_I)$, and functions $B_I(\sigma_{II})$ and $B_{II}(\sigma_I)$, such that $A_I(\sigma_{II})$ maintains a fixed, but distinct value for each characteristic Γ_I , and $B_I(\sigma_{II})$ maintains a fixed value for all characteristics Γ_I ; and $A_{II}(\sigma_I)$ and $B_{II}(\sigma_I)$ each maintain fixed, but distinct values for each characteristic Γ_{II} . In this case, we have the following correspondences on the characteristics of the expansion wave in domain $\mathscr{D}$:

Table 17

Characteristic	**Equation in Domain $\mathscr{D}$**	**Riemann Invariant**
Γ_I:	$x - \int (U+C)\cdot dt = A_I(\sigma_{II})$	$\frac{2}{\gamma-1}\cdot C + U = B_I(\sigma_{II})$
Γ_{II}:	$x - (U-C)\cdot t = A_{II}(\sigma_I)$	$\frac{2}{\gamma-1}\cdot C - U = B_{II}(\sigma_I)$

It is obvious that the Type I characteristics span the wave; hence $B_I(\sigma_{II})$ is determinate at plane 0-0, the plane of choked flow, as well as having a fixed value everywhere in $\mathscr{D}$. Note that at plane 0-0, we may write:

$$C \equiv U \equiv U_{cr} \tag{92}$$

In view of Equation (92) and the definition of $B_I(\sigma_{II})$, we find that:

$$B_I(\sigma_{II}) = \sqrt{\frac{2\gamma\cdot(\gamma+1)\cdot RT_0}{(\gamma-1)^2}} \tag{93}$$

Let e be the total energy of the flow at a point m in domain $\mathscr{D}$ and let $p\cdot v$ represent the flow work required to compress the flow adiabatically from its thermodynamic state at limit velocity to its actual thermodynamic state at point m . Also, let $p_s\cdot v_s$ represent the flow work to adiabatically stagnate the flow from that same thermodynamic state at limit velocity. Continuing, let u_s represent the local stagnation internal energy, and note that $e_s = u_s$. Now the First Law of Thermodynamics requires the change in total energy between these two states, at point m , in our adiabatic system, to be a consequence of the difference in flow work required to reach each of them from their common thermodynamic state at limit velocity. This condition is expressed by:

$$(e_s - e) + (p_s\cdot v_s - p\cdot v) = 0 \tag{94}$$

Let T_s represent the local stagnation temperature, and let T_z represent the local static temperature. Introducing the Perfect Equation of State and, the assumption that the gas is calorically perfect, Equation (94) becomes:

$$\frac{1}{(\gamma-1)}\cdot R(T_s - T_z) - \frac{U^2}{2} + R(T_s - T_z) = \frac{\gamma}{(\gamma-1)}\cdot R(T_s - T_z) - \frac{U^2}{2} = 0 \tag{95}$$

Under the assumed gas properties, we may write:

$$C = \sqrt{\gamma R T_z} \tag{96}$$

It immediately follows from Equation (95) that at every point in $\mathscr{D}$:

$$U^2 = \frac{2}{(\gamma - 1)} \cdot \left[\gamma R T_s - C^2\right] \tag{97}$$

Let us turn now to the Riemann Invariant on characteristics of Type I; and recall that it has a fixed value everywhere in $\mathscr{D}$, determined by Equation (93), as a result of the Type II characteristics being straight lines. We find that:

$$U^2 = B_I(\sigma_{II})^2 - \frac{4}{(\gamma - 1)} \cdot B_I(\sigma_{II}) \cdot C + \frac{4}{(\gamma - 1)^2} \cdot C^2 \tag{98}$$

Hence the stagnation temperature T_s may be written in terms of Riemann Invariant

$B_I(\sigma_{II})$ and local sound speed C at every point in $\mathscr{D}$. Equations (97) and (98) imply that:

$$T_s = \frac{1}{\gamma R} \cdot \left[(\gamma - 1) \cdot \frac{B_I^2(\sigma_{II})}{2} - 2C \cdot B_I(\sigma_{II}) + \left(\frac{\gamma + 1}{\gamma - 1}\right) \cdot C^2 \right] \tag{99}$$

Differentiating this expression with respect to sound speed, we find that:

$$\frac{dT_s}{dC} = \frac{2}{\gamma R} \cdot \left[\left(\frac{\gamma + 1}{\gamma - 1}\right) \cdot C - B_I(\sigma_{II}) \right] \tag{100}$$

Noting Equations (91), (92), and (93), we find that $\frac{dT_s}{dC} = 0$ at plane 0-0 and

decreases monotonically with decreasing C. Hence, Equation (100) implies that T_s^*, the

supremum of T_s, is located at the back face of shock $\mathscr{S}$ seen in Figure 7, and propagates with it; and T_{s*}, the infimum of T_s, is permanently located at the plane of choked flow. It follows from Equations (91), (92), (93), and (99) that:

$$T_{s*} = T_0 \tag{101.i}$$

$$T_s^* = \left(\frac{\gamma+1}{\gamma-1}\right)\cdot T_0 \tag{101.ii}$$

Equation (99) establishes that T_s is a strict function of C on Type I characteristics. Our problem is to calculate the mapping of T_s on domain $\mathscr{D}$ in the (x, t) plane. Consider Figure 6 again and take account that in the case of the backward-facing, simple wave, the roles of the Γ_I and Γ_{II} characteristics are reversed from those in the mapping portrayed there. For the backward-facing expansion wave, every characteristic Γ_I maps to the same single curve R in the (ρ ,U) plane. In consequence, $B_I(\sigma_{II})$, as defined in Table 17, is fixed for every point on R. Now note that selection of a point p in $\mathscr{D}$ establishes a unique σ_I and σ_{II} on a characteristic Γ_I, and thus establishes a point $\bar{p}$ on R. Clearly, this is a unique point in the (ρ ,U) plane; hence both ρ and U are uniquely determined. We have simultaneously uniquely determined C, as can be seen by utilization of the fact that $B_I(\sigma_{II})$ is a constant function on $\mathscr{D}$; but then Equation (99) may be used to establish a unique value of T_s which corresponds only to our unique value of C.

Continuing, the unique σ_I at point p establishes the fixed value for $A_{II}(\sigma_I)$ corresponding to the particular Γ_{II} passing through p, upon which U and C are fixed, since a Type II characteristic is a straight line in $\mathscr{D}$. Hence $f \restriction \Gamma_{II} = \bar{p}$ for Γ_{II} passing through p in $\mathscr{D}$. It is at once clear that selection of point p in $\mathscr{D}$ establishes $f \restriction \Gamma_{II}$ in its entirety for Γ_{II} passing through point p.

Letting C_* and ρ_* be, respectively, the sound speed and gas density at the plane of choked flow, we note that these quantities are fixed for all time. But the isentropic constraint implies that C and ρ at point $\bar{p}$ are related to C_* and ρ_* according to:

$$\rho = \rho_*\cdot\left(\frac{C}{C_*}\right)^{\frac{2}{\gamma-1}} \tag{102}$$

The preceding equation shows that the value of C corresponding to point $\bar{p}$ on R is in one-one correspondence with ρ at that point. Defining T as the mapping of domain $\mathscr{D}$ to the set of (T_s , U) pairs[1] corresponding one-one with the (ρ ,U) pairs comprising curve R, we have the following material equivalence:

$$f \upharpoonright \Gamma_{II} \equiv \mathrm{T} \upharpoonright \Gamma_{II} \tag{103}$$

By Equation (102) we have thus shown that $\mathrm{T} \upharpoonright \Gamma_{II}$ is also established in its entirety in $\mathscr{D}$ for Γ_{II} passing through point p . Since $\mathrm{T} \upharpoonright \Gamma_{II}$ is known at plane 0-0, the plane of choked flow, and is stationary there, we assert that the properties of the simple - wave mapping:

$$f : \mathscr{D} \rightarrow R$$

permit the exact calculation of the entire stagnation temperature mapping on $\mathscr{D}$.

Note that this wave is centered at point (0, 0) in the (x, t) plane, and let us refer to this point as θ . This means that every characteristic Γ_{II} in domain $\mathscr{D}$ must have point θ as a limiting point. Inspection of Table 17 shows that $A_{II}(\sigma_I) = 0$ everywhere in $\mathscr{D}$, and hence is known a priori for every value of σ_I in $\mathscr{D}$, as a result.

In order to construct the mapping:

$$\mathrm{T} : \mathscr{D} \rightarrow R$$

we need to know the stagnation temperature, T_0 , at plane 0-0. If we wish to construct f in conjunction with T , the stagnation pressure, p_0 , must be known at plane 0-0, as well. Given these quantities, let us select any T_s such that $T_{s*} \leq T_s < T_s$ and solve the following quadratic equation for C:

$$\left(\frac{\gamma+1}{\gamma-1}\right) \cdot C^2 - 2C \cdot B_I(\sigma_{II}) + (\gamma-1) \cdot \frac{B_I^2(\sigma_{II})}{2} - \gamma R \cdot T_s = 0 \tag{104}$$

[1] Refer to Equation (99).

The root for which $\frac{dT_s}{dC} \leq 0$, as determined from Equation (100), is the unique value for C corresponding to T_s for every (x, t) pair in $\mathscr{D}$ mapping to T_s . Now $B_I(\sigma_{II})$ is determinate from Equation (93); hence U is determinate from the Riemann Invariant in the first row of Table 17. This establishes the appropriate point $\bar{p}$ in the range of T . By Equation (102), point $\bar{p}$ in the range of f is also established. One can see that there exist unique $U(\bar{p})$ and $C(\bar{p})$ such that there is a family $\{L_\alpha\}_{\alpha=1,2,\ldots}$ of lines expressed as:

$$L_\alpha \equiv \left\{ x - \left(U(\bar{p}) - C(\bar{p}) \right) \cdot t \ = \{Const\}_\alpha \right\}$$

and such that $f \restriction \{L_\alpha\}_{\alpha=1,2,\ldots} = \bar{p}$. Obviously, there exists α such that $L_\alpha = \Gamma_{II}$, where Γ_{II} is the unique characteristic of Type II mapping to point $\bar{p}$, for some member of $\{L_\alpha\}_{\alpha=1,2,\ldots}$. In our case, $\{Const\}_\alpha = A_{II}(\sigma_I) = 0$; and since Γ_{II} corresponding to point $\bar{p}$ must be limited at θ , we conclude that it is completely described in domain $\mathscr{D}$ by:

$$x - \left(U(\bar{p}) - C(\bar{p}) \right) \cdot t \ = 0 \qquad \textbf{(105)}$$

Thus, when T_s has been established, one may establish a point $\bar{p}$ in the range of function, T ; and thence establish the entire mapping $\mathrm{T} \restriction \Gamma_{II} = \bar{p}$. Since we are free to pick any T_s such that $T_{s*} \leq T_s < T_s^*$, and since the entire mapping:

$$\mathrm{T} : \mathscr{D} \rightarrow R$$

is the union of mappings $\mathrm{T} \restriction \Gamma_{II}$, a procedure for the exact calculation of the mapping of T_s on domain $\mathscr{D}$ is established.

It is not necessary for the calculation of the exact mapping, T , to establish the

trajectories of the Γ_I characteristics. However, we may wish to establish these trajectories in the (x, t) plane in order to visualize the properties of $\mathbf{T}$. Consider point o in Figure 8, and let us suppose that T_s is known exactly there. This means that U and C are exactly known there, as well, since $B_I(\sigma_{II})$ is known and fixed at every point in $\mathscr{D}$[1]. Suppose that U and C are known for another Γ_{II} characteristic in domain $\mathscr{D}$.

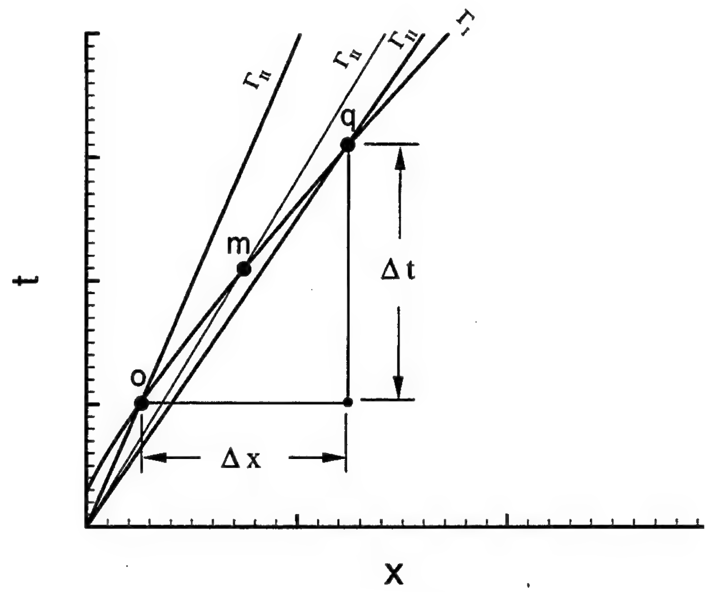


Figure 8

Our problem is to calculate the coordinates of the intersection of the new Γ_{II} characteristic with the Γ_I characteristic passing through point o.

Recalling that σ_I parametrizes Γ_I, and letting $\Delta\sigma_I$ be the difference between the values of σ_I at points o and q, we note that the Mean Value Theorem guarantees the existence of a point m on Γ_I between o and q such that:

$$x(q) - x(o) = \left[U(m) + C(m)\right]\cdot\left[t(q) - t(o)\right] \tag{106}$$

But $B_I(\sigma_{II})$ is fixed throughout $\mathscr{D}$. Hence, Equation (106) becomes:

$$x(q) - x(o) = \left[B_I(\sigma_{II}) + \left(\frac{\gamma - 3}{\gamma - 1}\right)\cdot C(m)\right]\cdot\left[t(q) - t(o)\right] \tag{107}$$

[1] Refer to Equations (100) and (104).

We shall assume that C is infinitely differentiable with respect to σ_I . In this case, there exists a real number, K, such that $0 \le K \le 1$, and such that the following Taylor's expansions hold:

$$C(o) = C(m) - \frac{C^{I}(m)}{1!} \cdot K\Delta\sigma_I + \frac{C^{II}(m)}{2!} \cdot K^2(\Delta\sigma_I)^2 - \frac{C^{III}(m)}{3!} \cdot K^3(\Delta\sigma_I)^3 + \cdots \tag{108}$$

and

$$C(q) = C(m) + \frac{C^{I}(m)}{1!} \cdot (1-K) \cdot \Delta\sigma_I + \frac{C^{II}(m)}{2!} \cdot (1-K)^2 \cdot (\Delta\sigma_I)^2 + \frac{C^{III}(m)}{3!} \cdot (1-K)^3 \cdot (\Delta\sigma_I)^3 + \cdots \tag{109}$$

Letting

$$C(p) = \frac{C(o) + C(q)}{2}$$

we may write:

$$C(p) = C(m) + \frac{C^{I}(m)}{2 \cdot 1!} \cdot (1-2K) \cdot \Delta\sigma_I + \frac{C^{II}(m)}{2 \cdot 2!} \cdot (1-2K+2K^2) \cdot (\Delta\sigma_I)^2 + \frac{C^{III}(m)}{2 \cdot 3!} \cdot (1-3K+3K^2-2K^3) \cdot (\Delta\sigma_I)^3 + \cdots \tag{110}$$

If $\bar{x}(q)$ is the predicted value of x at point q using $C(p)$ in place of the correct mean value for C, Equation (107) implies that:

$$\bar{x}(q) = x(o) + \left[B_I(\sigma_{II}) + \left(\frac{\gamma-3}{\gamma-1}\right) \cdot C(p)\right] \cdot \left[t(q) - t(o)\right] \tag{111}$$

Apparently, Equations (107) and (111) imply that:

$$\bar{x}(q) - x(q) = \left(\frac{\gamma-3}{\gamma-1}\right)\cdot\left[\frac{C^{I}(m)}{2\cdot 1!}\cdot(1-2K)\cdot\Delta\sigma_I + \frac{C^{II}(m)}{2\cdot 2!}\cdot(1-2K+2K^2)\cdot(\Delta\sigma_I)^2 + \right.$$

$$\left.\frac{C^{III}(m)}{2\cdot 3!}\cdot(1-3K+3K^2-2K^3)\cdot(\Delta\sigma_I)^3 + \cdots\right]\cdot\left[t(q) - t(o)\right] \tag{112}$$

We note that $\left|t(q) - t(o)\right| \le \left|\Delta\sigma_I\right|$. This inequality implies that:

$$\left|(\Delta\sigma)^{n-1}\cdot\left[t(q) - t(o)\right]\right| \le \left|(\Delta\sigma_I)^n\right|$$

Consequently, the following inequality holds.

$$\left|\bar{x}(q) - x(q)\right| \le \left|\frac{\gamma-3}{\gamma-1}\right|\cdot\left|\left|\frac{C^{I}(m)}{2\cdot 1!}\cdot(1-2K)\cdot(\Delta\sigma_I)^2\right| + \left|\frac{C^{II}(m)}{2\cdot 2!}\cdot(1-2K+2K^2)\cdot(\Delta\sigma_I)^3\right| + \right.$$

$$\left.\left|\frac{C^{III}(m)}{2\cdot 3!}\cdot(1-3K+3K^2-2K^3)\cdot(\Delta\sigma_I)^4\right| + \cdots\right| \tag{113}$$

The lead term of Equation (113) shows that the use of the average value for sound speed C on an interval of Γ_I in $\mathscr{D}$ results in a second order error, expressed as $O(\Delta\sigma_I)^2$, in $x(q)$.

If the coordinates of point o in $\mathscr{D}$ are known, and T_s is known there; and T_s can be established at some other point q on Γ_I passing through point o, then the coordinates of q in $\mathscr{D}$ can be established to $O(\Delta\sigma_I)^2$. Note that the Γ_{II} characteristics are centered at the *origin of coordinates*. Since they are straight lines in $\mathscr{D}$, we may write:

$$x(q) = \left[U(q) - C(q)\right]\cdot t(q) \tag{114}$$

Since $B_I(\sigma_{II})$ is fixed throughout $\mathscr{D}$, Equation (114), which is exact, becomes:

$$x(q) = \left[B_I(\sigma_{II}) - \left(\frac{\gamma+1}{\gamma-1}\right)\cdot C(q)\right]\cdot t(q) \tag{115}$$

Let $\Delta t = t(q) - t(o)$ and $\Delta x = x(q) - x(o)$. Then, to within $O(\Delta \sigma_I)^2$, the following pair of equations exists:

$$x(q) = x(o) + \Delta x \tag{116.i}$$

$$t(q) = t(o) + \frac{\Delta x}{\left[B_I(\sigma_{II}) + \left(\frac{\gamma - 3}{\gamma - 1}\right) \cdot C(p)\right]} = \frac{x(q)}{\left[B_I(\sigma_{II}) - \left(\frac{\gamma + 1}{\gamma - 1}\right) \cdot C(q)\right]} \tag{116.ii}$$

If point $\left(x(o),\, t(o)\right)$ is established on a Γ_I characteristic, then Equations (116.i) and (116.ii) establish point $\left(x(q),\, t(q)\right)$ on that same characteristic to within $O(\Delta \sigma_I)^2$.

7. RESULTS FROM EXAMPLE CALCULATION FOR AN INFINITE RESERVOIR SUSTAINING STAGNATION PRESSURE OF 65.0 PSIA AND STAGNATION TEMPERATURE OF 740 °R AT PLANE 0-0

Let us proceed with the calculation of a backward-facing expansion wave issuing into a void main connected to an infinite reservoir containing standard air, and maintaining a stagnation pressure of 65.0 psia, and a stagnation temperature of 740 ° R at plane 0-0, the plane of choked flow. A characteristic net on domain $\mathscr{D}$ in the (x, t) plane for this system, represented by Equations (27.i) and (27.ii), is illustrated in Figure 9 on the following page. The initial and boundary conditions are given at point (0, 0) in $\mathscr{D}$; with $x = 0$ being located at the plane 0-0. The line (0, t) defines the lead Γ_{II} characteristic in $\mathscr{D}$. The thermodynamic state is constant on this line since constant stagnation pressure and temperature are perpetually maintained at plane 0-0. Hence, $J = 0$ throughout $\mathscr{D}$, and U and C are each fixed on each Γ_{II} characteristic. It follows that each Γ_{II} characteristic is a straight line in the (x, t) plane; but each Γ_{II} characteristic carries a different thermodynamic state, and has a different slope. The wave is centered at (0, 0), i.e., the characteristics all emanate from (0, 0). Thus, the example wave is a centered, backward-facing, simple rarefaction wave. The Γ_I characteristics span the wave, and fail to be straight lines. However, they may be calculated to within $O(\Delta \sigma_I)^2$ as explained in Section 6.

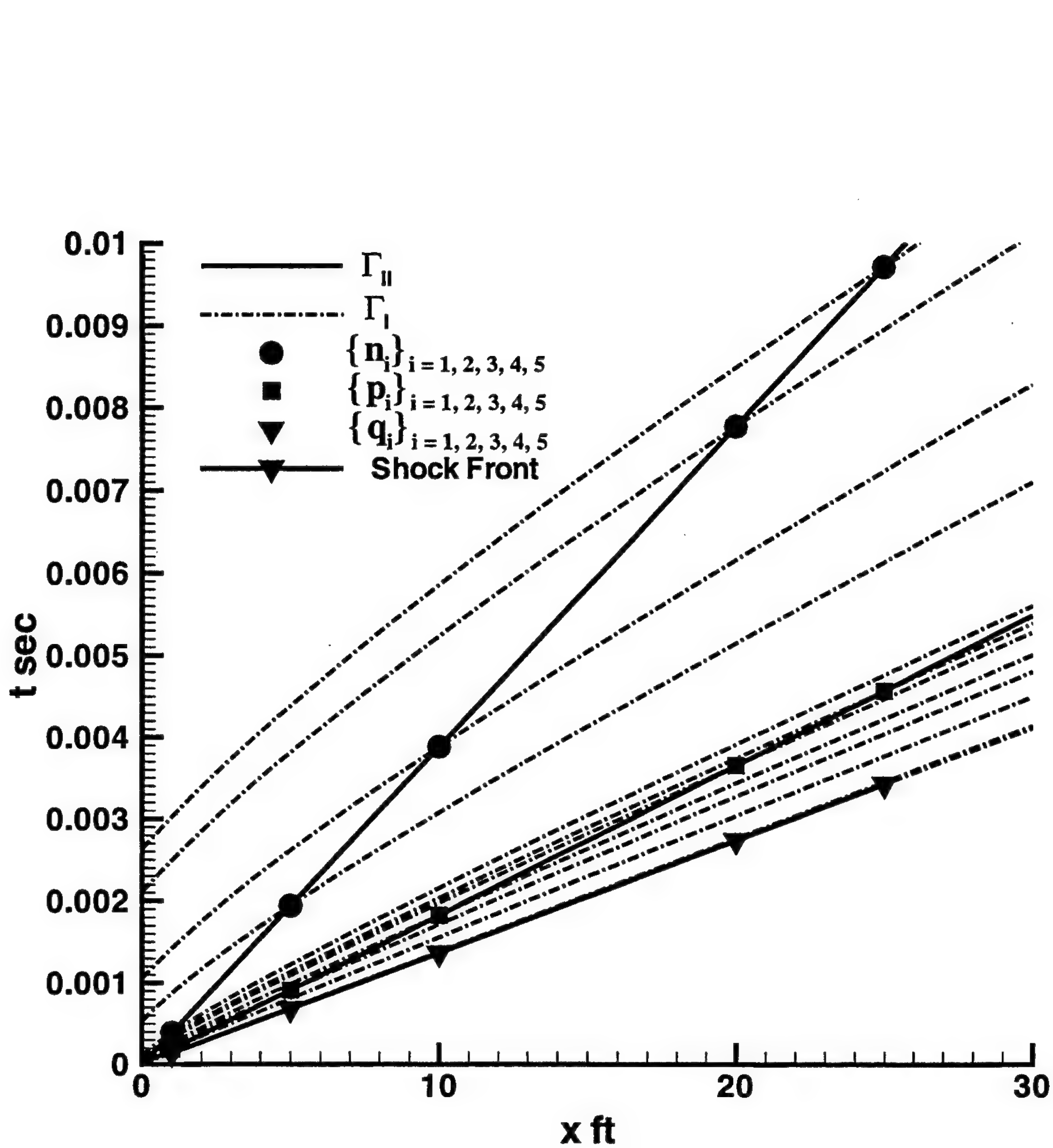


Figure 9

We re-emphasize that it is not necessary to calculate the trajectories of the Γ_I characteristics in order to establish function f, or its material equivalent, the temperature mapping $\mathbf{T}$.

There are three solid lines illustrated in Figure 9 which are characteristics of type Γ_{II} in the domain of the example wave. Each of these characteristics is marked with five points from domain $\mathscr{D}$ of the wave. The points associated with the solid circles in Figure 9 are designated with $\{n_i\}_{i=1,2,3,4,5}$, and the points associated with the squares are designated with $\{p_i\}_{i=1,2,3,4,5}$, . . . , etc., where the index increases with increasing x and t. When the

shock front $\mathscr{S}$ is approached in domain $\mathscr{D}$, the Γ_I and Γ_{II} characteristics become coincident in the limit. The solid line marked with the inverted triangles represents that limit.

Now the range of f on $\mathscr{D}$, for the example wave, is the single curve, R, in the (ρ, U) plane - illustrated in Figure 10. Considering the developments of Section 5, it is clear that f has no inverse. All of the points on a given characteristic map to a single point on the curve. For example, all five domain points indicated by solid circles in Figure 9 map to the point associated with the single solid circle, designated as $\bar{n}$, on the curve in Figure 10. The lead characteristic coincides with the ordinate passing through (0, 0), and is not marked in Figure 9. However, the range point to which it maps is illustrated as the hollow circle labelled 0 in Figure 10.

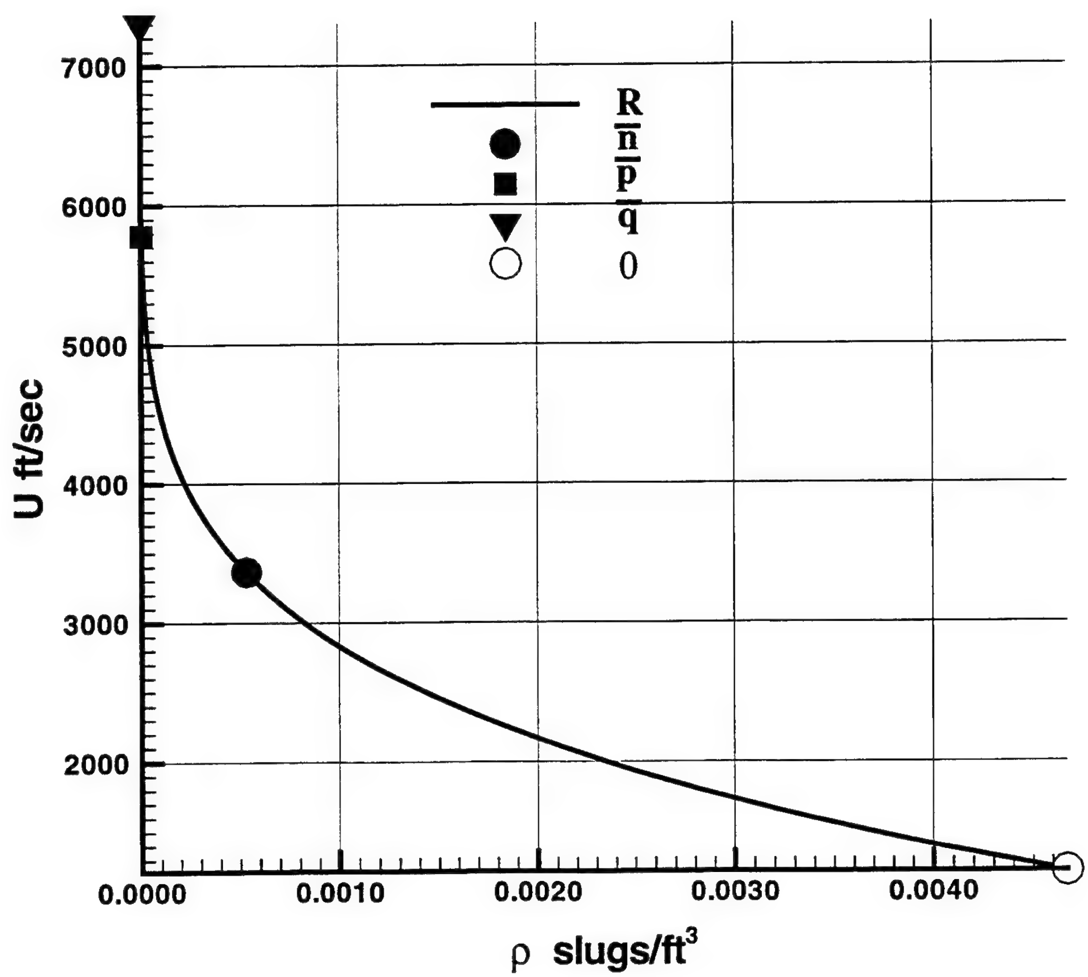


Figure 10

We observe that escape velocity, U_L , discussed on page 42, is proportional to U_{cr} at plane 0-0, given by Equation (91). The relationship may be written as:

$$U_L = B_I(\sigma_{II}) = \left(\frac{\gamma+1}{\gamma-1}\right)\cdot U_{cr} \tag{117}$$

In Figure 10, the inverted triangle denotes the limit of characteristics. One can deduce from its position that in our example wave, $U_L = 7303.4726\ ft/\text{sec}$ while $U_{cr} = 1217.33\ ft/\text{sec}$. The stated magnitudes of these velocities are in precisely the ratio given by Equation (117) when γ for standard air is incorporated.

It is clear that the limit velocity at plane 0-0 is substantially less that escape velocity for $\mathscr{D}$. The equation for limit velocity at plane 0-0 in terms of both $B_I(\sigma_{II})$ and U_{cr} is given as follows:

$$U_{l0} = \sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)}\cdot B_I(\sigma_{II}) = \sqrt{\left(\frac{\gamma+1}{\gamma-1}\right)}\cdot U_{cr} \tag{118}$$

Incorporating γ for standard air, we find that $U_{l0} = 2981.84\ ft/\text{sec}$ in our example wave. One can see that U_L is larger than U_{cr} by a factor of 2.4495 - the effect being due to the flow work done to form the shock at $\mathscr{I}$ at the time of initiation of the expansion wave.

Noting that f symbolizes the subject wave function, we have shown that under the hypotheses laid down in Section 3, the representation of the mapping:

$f : \mathscr{D} \to R$

given from characteristic theory is exact. We have also shown that on each Γ_{II} characteristic, $f \restriction \Gamma_{II} \equiv T \restriction \Gamma_{II}$. This means that if one were to select a point in $\mathscr{D}$, such as, $\{p_i\}_{i=1}$ from Figure 9, the unique pair, (T_s, U) , which corresponds to it may be

calculated precisely and directly. Since the Γ_{II} characteristics are straight and centered at (0, 0), there exists a constant, A, such that $0 \leq A < B_I(\sigma_{II})$, and such that the following equation pair holds in $\mathscr{D}$:

$$\frac{x}{t} = -C + U = A \tag{119.i}$$

$$\frac{2}{\gamma-1}\cdot C + U = B_I(\sigma_{II}) \tag{119.ii}$$

Now $B_I(\sigma_{II})$ is constant throughout $\mathscr{D}$, and determinate from the given stagnation temperature at plane 0-0, using Equation (93) from Section 6. For the selection of a point in $\mathscr{D}$, such as $\{p_i\}_{i=1}$, it is clear that A is determined by $\frac{x}{t}$ for that point. Hence C and U are each uniquely determined by the following equation pair:

$$C = \left(B_I(\sigma_{II}) - A\right)\cdot\left(\frac{\gamma-1}{\gamma+1}\right) \tag{120.i}$$

$$U = \left(B_I(\sigma_{II}) + \frac{2}{\gamma-1}\cdot A\right)\cdot\left(\frac{\gamma-1}{\gamma+1}\right) \tag{120.ii}$$

The stagnation temperature, T_s, is immediately determinate by means of Equation (99) from Section 6, as a result.

Under the mapping, T, there is a unique pair, (T_s, U), corresponding to each choice of (x, t) in $\mathscr{D}$. Hence T_s is a function of x and t, and our wave may be represented in this fashion. Note that in view of the material equivalence, $f \restriction \Gamma_{II} \equiv \mathrm{T} \restriction \Gamma_{II}$, and the

condition that $J = 0$, every point on the Γ_{II} characteristic passing through a given point in $\mathscr{D}$ maps to the same value of T_s. If our choice of (x, t) had been point $\{p_i\}_{i=1}$ in Figure 9, for instance, then the points, $\{p_i\}_{i=1,2,3,4,5}$, represented by the solid squares, as well as any of the other points on the Γ_{II} characteristic passing through $\{p_i\}_{i=1}$, would map to the same value, $T_s = 2820^0 R$.

The wave function solving our example problem is represented as $T_s(x, t)$ in Figure 11, seen on the following page. Each contour label is the distance, x, in feet, from the plane of choked flow, i.e., plane 0-0. Again, we note that this representation is exact. Hence, Figure 11 is the representation of the function $T_s(t)$ at each of the indicated distances from plane 0-0. Stagnation temperature, T_s, depending upon $\{p_i\}_{i=1,2,3,4,5}$ in Figure 9, is the set of points represented by the solid squares in Figure 11, which are sequenced from left to right with increasing index i. The common value, $T_s = 2820^0 R$, which applies at each of the points, $\{p_i\}_{i=1,2,3,4,5}$, appears on each of the five contours to which each of these points corresponds.

Referring to Equation (119.i), one can see that constant A is the propagation velocity of a particular point in the range of the wave carrying a particular value of stagnation temperature T_s. In our example case, we directly calculate that $A = 5476.8592\ ft/\text{sec}$ for the Γ_{II} characteristic marked by points $\{p_i\}_{i=1,2,3,4,5}$ in Figure 9. Employing Equations (120.i) and (120.ii), and noting that $B_I(\sigma_{II}) = 7303.4726\ ft/\text{sec}$, we find that $C = 304.43557\ ft/\text{sec}$ and $U = 5781.29468\ ft/\text{sec}$. Turning to Equation (99), we calculate that $T_s = 2820.684^0 R$, which is sufficiently close to the previously determined

value of $2820^0 R$. This calculation of course, can be performed at any point in $\mathscr{D}$.

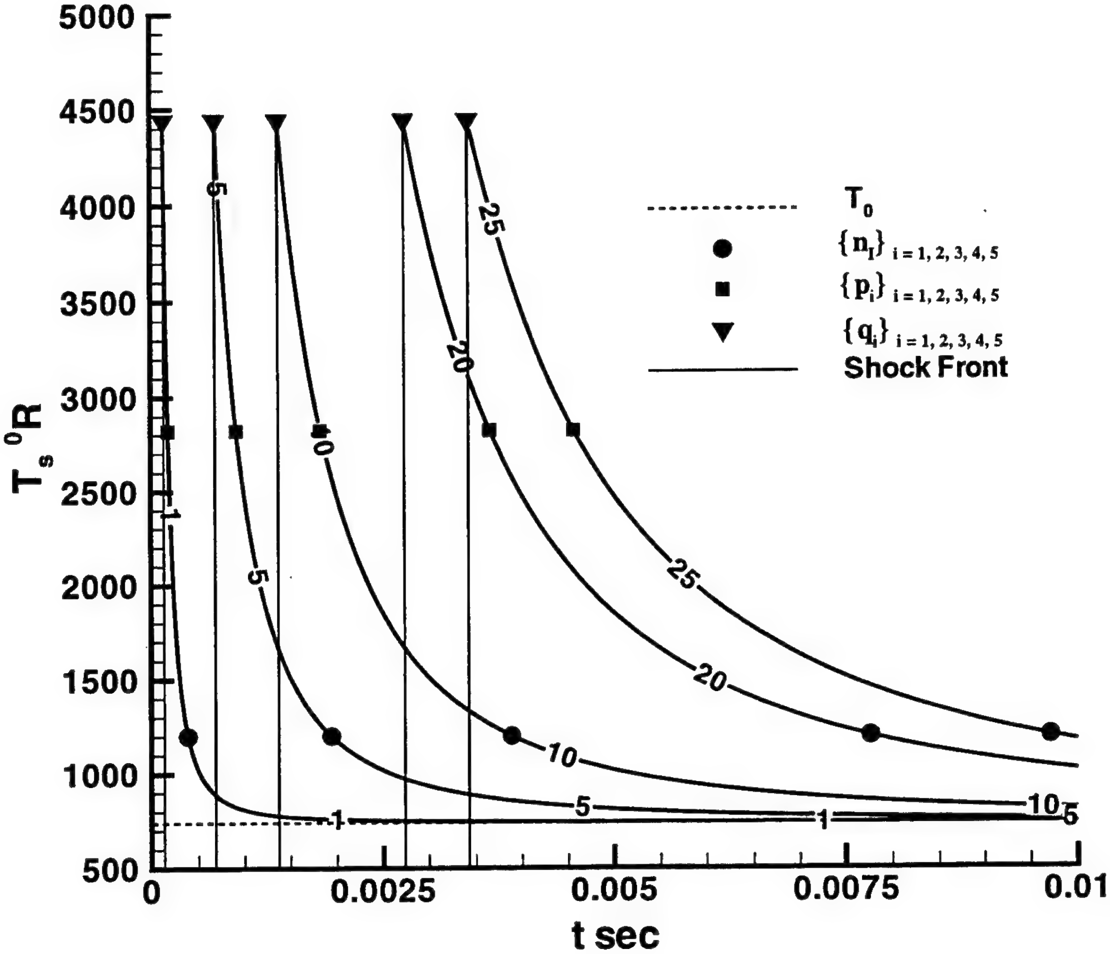


Figure 11

It is clear that each value of T_s in the range of our wave has a different velocity of propagation; the range of velocities being $0 \le A < B_I(\sigma_{II})$. The value of T_s at the critical flow point, i.e., plane 0-0, is given by Equation (101.i). At this location, $T_s = T_{s*}$. However, the value of T_s at the rear face of the shock is $6 \times T_0$ in standard air

according to Equation (101.ii). Here, we write: $T_s = T_s^*$. The extreme temperatures in the range of our wave, as well as that given for our example point $\bar{p}$, along with their velocities of propagation are given in the following table:

	T_s	*Velocity of Propagation*
Shock	4440 ^{0}R	7303.47260 ft/sec
Point $\bar{p}$	2820 ^{0}R	5476.8592 ft/sec
Critical Point	740 ^{0}R	0.0 ft/sec

It is at once apparent that the critical point in the range of the wave, represented by the last line of the table, has no propagation velocity at all. It does not advance into the infinite main. It is on the lead characteristic of the wave, and is permanently located at plane 0-0 from initial time forward.

One can see by examination of Figure 11 that the shape of $T_s(t)$ is different for each of the five linear distances from plane 0-0 indicated by the contour labels. This is the result of each part of the wave having a different propagation velocity. The distribution of $T_s(t)$, at each distance, is an artifact of the initial flow work done to create the shock which proceeds into the vacuum of the main. The vertical lines mark the time of appearance of the shock, following initiation of the wave at time zero, at each of the five indicated distances from plane 0-0.

Let us turn, now, to the static pressure function, $p(x,t)$. At each distance x from plane 0-0, $p = 0$ at initial time, and asymptotically approaches a limit given by the following formula:

$$p_{cr} = p_0 \cdot \left[\left(\frac{2}{\gamma-1} \right) \cdot \left(\frac{U_{cr}}{U_{l0}} \right)^2 \right]^{\frac{\gamma}{\gamma-1}} = p_0 \cdot \left[\left(\frac{2}{\gamma+1} \right) \right]^{\frac{\gamma}{\gamma-1}} \qquad \textbf{(121)}$$

where U_{cr} and U_{l0} are precisely given by Equations (91) and (118), respectively. This

limit exists due to U_{cr} being the lower limit of velocity for domain $\mathscr{D}$. For our example case, we calculate p_{cr} = 34.3381 *psia* from Equation (121).

Figure 12 shows $p(t)$ at each of the same five distances from plane 0-0 given by the contour labels in Figure 11. Those distances are represented by the contour labels in Figure 12, just as they were in Figure 11. The indicated points on these contours are the images of the similarly marked points on characteristics shown in Figure 9; as was the case for the T_s points in the range of $T(t)$ shown in Figure 11. The grad symbol was made hollow in this plot so that the image points, $\{q_i\}_{i=1,2,3,4,5}$, could be represented despite the image positions being so compacted due to the relatively gradual rate of rise of static

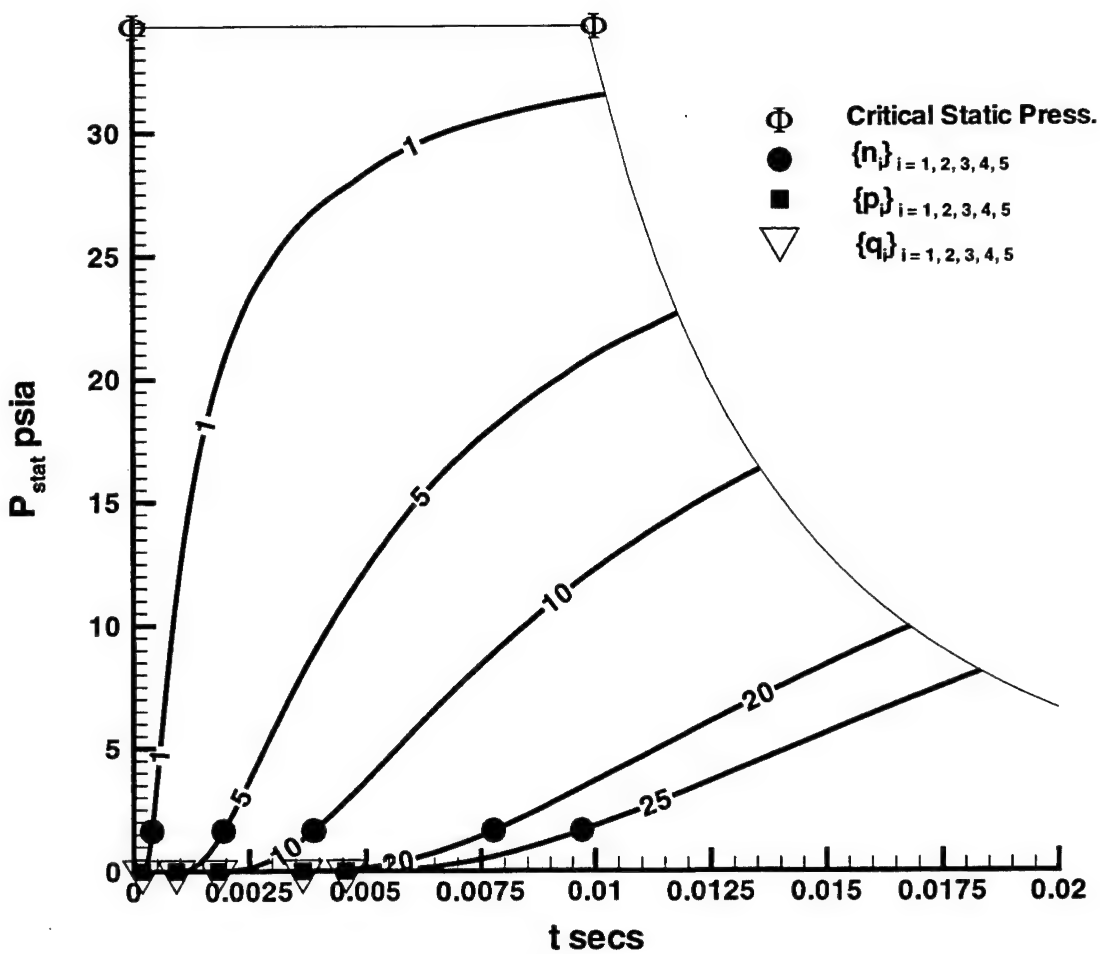


Figure 12

pressure. The static pressure limit, i.e., critical static pressure is denoted by Φ .

On the interval over which the analysis was performed, Figure 12 clearly shows the asymptotic behavior of $p(t)$ at one foot distance from plane 0-0; although, even at the one-foot distance, it has only achieved 91.7% of its limit value after 10 milliseconds.

The variation in velocity of propagation throughout $\mathscr{D}$ has considerable effect on the form of $p(t)$ as distance from plane 0-0 increases. The form of $p(t)$ at 25 feet from plane 0-0 is dissimilar to its form at 1 foot distance - its approach to the limit being much more gradual at 25 feet, as can be seen from the lower right-hand contour in Figure 12.

Figure 13 represents $T_s(x,t)$ as $T_s(x)$ at various times after initiation of the wave at plane 0-0. In this figure, the contour labels represent the time in seconds after the

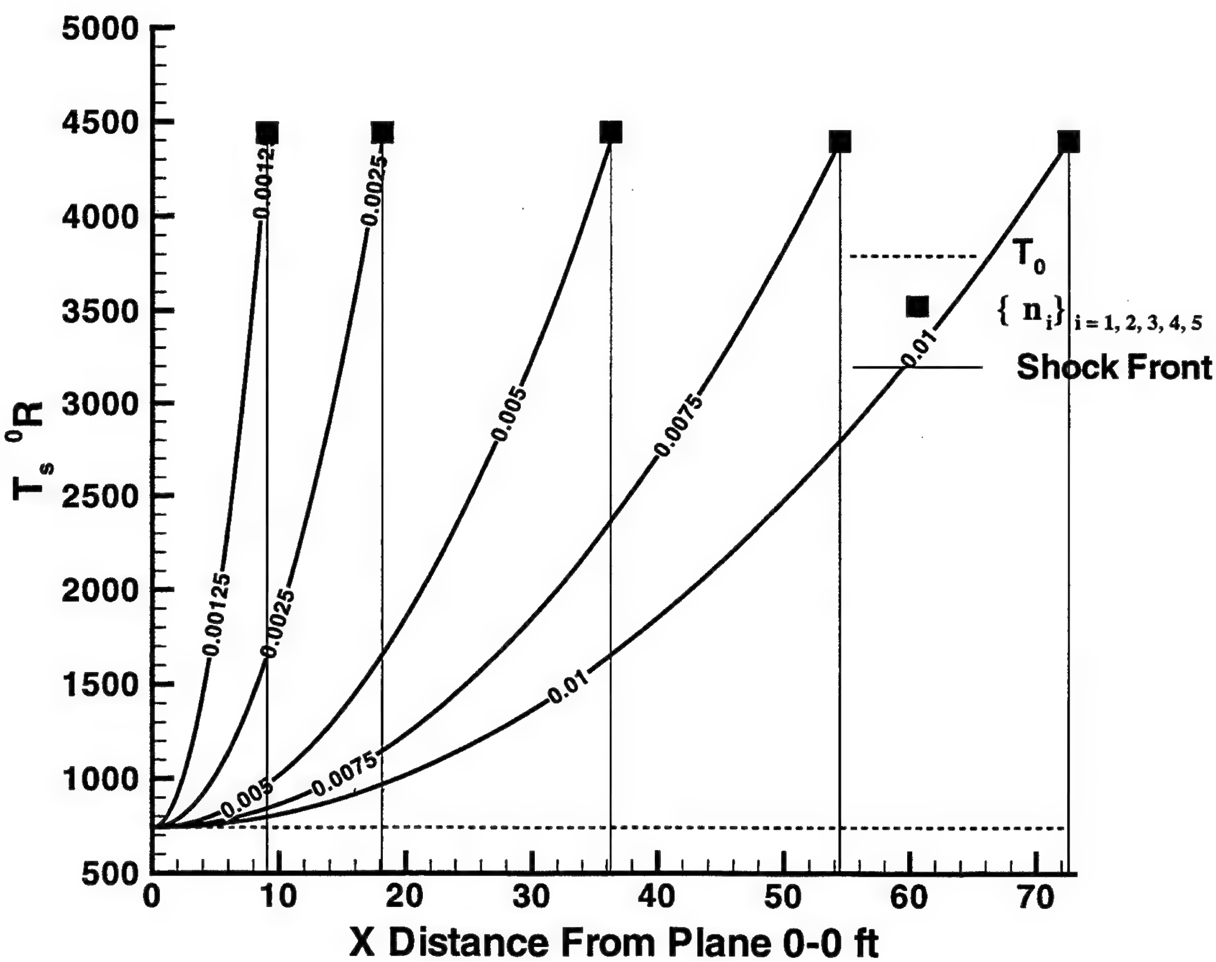


Figure 13

initiation of the expansion wave, while the contours, themselves, represent the distribution of stagnation temperature. In this figure, the vertical lines give the shock position at each of the times indicated by the contour labels. The solid squares indicate the stagnation temperature immediately behind the shock at each of these times.

The stagnation temperature immediately behind the shock retains the identical value of 4440^0 R at each of the shock positions shown in Figure 13. However, it is clear that the stagnation temperature decreases at each location after the passage of the shock, eventually approaching the lower limit of 740^0 R marked by the dotted line. At each distance, x, and at this limit, the flow velocity is precisely critical.

Finally, the stagnation temperature in the critical flow existing at plane 0-0 is the result of the work of expansion to accelerate the fluid from its quiescent condition in the reservoir to its state at plane 0-0. This acceleration is accomplished by an isentropic expansion wave which propagates into the subsonic flow in the reservoir, in the opposite direction from the rarefaction wave in the main. The result is that the stagnation temperature and pressure at plane 0-0 are each *always* less than the stagnation temperature and pressure in the reservoir. For our reservoir, which extends from the origin to $-\infty$ on the x-axis, and is infinite in extent in every direction perpendicular to the x-axis, and in which the flow is one-dimensional, this expansion wave is confined to plane 0-0 and, in this limiting case, becomes a *stationary, isentropic expansion shock*[1]. One can deduce that the Second Law of Thermodynamics is not contradicted in this instance[2].

The stagnation pressure and temperature at plane 0-0 are maintained, respectively, at $P_0 = 65.0\ psia$ and $T_0 = 740.0^0 R$ after initial time in our example problem. Under this condition, the corresponding static pressure and static temperature at plane 0-0 are each $p_1 = 34.338\ psia$ and $T_1 = 616.667^0 R$. The expansion wave, which must be present at plane 0-0, requires that the stagnation pressure and temperature in the reservoir remain at $p_{RO} = 82.419\ psia$ and $T_{RO} = 791.940^0 R$, respectively, for $t \geq 0$, in order to maintain the specified values of p_0 and T_0 at plane 0-0.

8. CONCLUSIONS

A one-dimensional analysis of the flow resulting from the expansion of a calorically perfect gas from an infinite reservoir into a void main implies that the rarefaction wave associated with this expansion is preceded by a shock propagating into the vacuum of the main at a rate given by Equation (117) on page 55. The argument leading to this conclusion is found in Section 5 on pages 40-42. This argument also implies that the velocity of propagation of the shock is the least upper bound of the velocity of propagation of a disturbance in the domain of the rarefaction wave. If the specific heat ratio for standard air is applied, then the propagation velocity of the shock is six times the critical velocity corresponding to the thermodynamic state at plane 0-0, seen in Figure 1, Section 3.

[1] The development of this model is given in the Appendix.
[2] See the development of Equations (A1), (A2), and (A3) in the Appendix, as a limiting case.

Concomitantly, the stagnation temperature immediately following the shock is given by Equation (101.ii). The argument leading to this equation is given in Section 6, pages 42-46. If the specific heat ratio for standard air is applied, the stagnation temperature following the shock is found to be six times the stagnation temperature at plane 0-0, as well.

The roles of the First and Second Laws of Thermodynamics in the development of the transient flow characterized by this wave are developed in Section 3, and lead to Equation (95) on page 44; and to Equations (99) and (100) on page 45 in Section 6. These equations, in turn, lead to the conclusion that the local stagnation temperature in the domain of the rarefaction wave is strictly dependent upon local sound speed; and increases as sound speed decreases, reaching its limit at the back face of the shock. This limit temperature does not change over time, and appears immediately following the shock at each downstream location in the main.

Apparently, the heat of compression, i.e., the flow work done to form the shock when the flow is initiated at plane 0-0, produces the thermal spike which one calculates from the application of characteristic theory to the flow interior to the rarefaction wave. Since the flow is inviscid in the analysis presented in this report, the thermodynamic process is reversible everywhere in the interior of the wave; hence, the calculated spike is not the result of frictional heating in the established flow. Even though the one dimensional and inviscid hypotheses represent substantial simplification of the actual physical flow found in the transient blow-down facility, they embody sufficient physics to imply that the thermal spike observed there is an artifact of the flow work done to establish the shock.

The one-dimensional hypothesis implies that the flow in the reservoir must be quiescent, i.e. in a state of rest, and therefore steady, even at the initiation of the rarefaction wave. This, in fact, is not true, even in an infinite reservoir. The flow in the neighborhood of the inlet to the main must have some radial component, and must be established by an expansion wave which proceeds into the reservoir, however diffuse that wave must be. This means that the stagnation temperature at plane 0-0, where the flow is critical, is actually less than the stagnation temperature in the quiescent state in the reservoir.

By insisting that the flow be one-dimensional in the infinite reservoir, one must admit the existence of an *isentropic expansion shock* at plane 0-0. It is shown in the appendix that the isentropic shock does not contradict the Second Law of Thermodynamics under the hypotheses of this problem; rather, it exists as the limiting case of the expansion wave which proceeds into the reservoir. Hence, the representation of the physics of the development of the backward-facing expansion wave is not fundamentally altered by the one-dimensional hypothesis. The application of characteristic theory to the two-dimensional equations of motion would allow one to characterize the expansion wave which proceeds into the reservoir, and therefore estimate its true effect.

REFERENCES

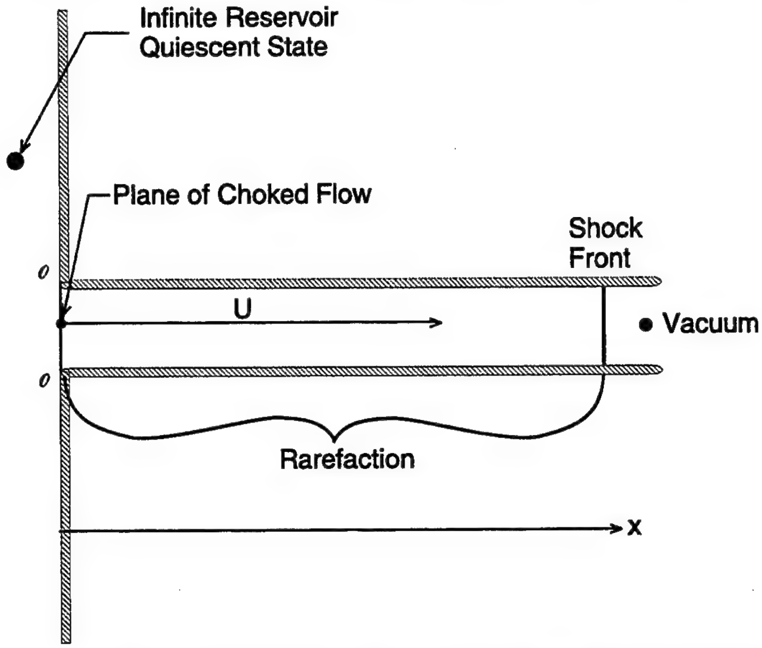
1. Bartle, Robert, *The Elements of Real Analysis*, John Wiley and Sons, Inc., New York, New York, 1964, pp 256-258

2. R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, Inc. New York, 1948, pp 92-107

3. Huang, Francis F., *Engineering Thermodynamics*, Macmillan Publishing Co., Inc., New York, 1976, pp 48-52

APPENDIX

Consider again the adiabatic, infinite reservoir and void main of Figure 1. The reservoir has infinite extent in the horizontal direction to the left of plane 0-0, and in any radial direction relative to its axis. Let us extend a cylindrical control volume, $\mathscr{L}$, of the same diameter as the main from plane 0-0 to any distance, L, to the left of plane 0-0. We suppose that the right face of $\mathscr{L}$ is coincident with plane 0-0, and that $\mathscr{L}$ is coaxial with the main. Let the following notation apply: σ_{0-0} represents the surface of $\mathscr{L}$ coinciding with plane 0-0, σ_+ represents the remainder of the surface of $\mathscr{L}$ when σ_{0-0} is excluded, and σ represents the entire surface of $\mathscr{L}$. Let $L_{-\infty}$ represent the length of $\mathscr{L}$ as L is extended to $-\infty$. Note that $L_{-\infty}$ does not represent a finite number, nor an attainable length. Finally, let subscript RO apply to the thermodynamic state in the infinite reservoir, and let subscript 1 apply to the thermodynamic state at plane 0-0.



Gas Expansion From Infinite Reservoir Into Vacuous Main

Now, the thermodynamic state is fixed in time throughout $\mathscr{L}$, and accumulated mass of gas is infinite when $L=L_{-\infty}$. Subsequently, if the flow is one-dimensional, the following conservation relations hold for the half-open time interval $0 \leq t < \infty$:

Momentum

$$\iint_{\sigma} [\vec{V}(\rho \vec{V}\cdot\hat{n}) + p\cdot\hat{n}]\,d\sigma = \left[p_{RO} - \left(p_1 + \rho_1 U_{cr}^2\right)\right]\cdot\sigma_{0-0} = 0 \qquad \text{(A1)}$$

Mass Flow

$$\iint_{\sigma} \rho \vec{V}\cdot\hat{n}\, d\sigma = \iint_{\sigma+} \rho \vec{V}\cdot\hat{n}\, d\sigma + \quad + \quad \rho_1 U_{cr} \cdot \sigma_{0\text{-}0} = 0 \tag{A2}$$

$$L = L_{-\infty} \Rightarrow \left\{ \vec{V}(\sigma_+) = 0 \wedge \rho(\sigma_+) = \rho_{R0} \right\} \; for \quad \rho_1 U_{cr} \cdot \sigma_{0\text{-}0} > 0$$

Entropy

$$\iint_{\sigma} s(\rho \vec{V})\cdot\hat{n}\, d\sigma = s_{R0} \cdot \iint_{\sigma+} \rho \vec{V}\cdot\hat{n}\, d\sigma + \quad + \quad s_1 \cdot \rho_1 U_{cr} \cdot \sigma_{0\text{-}0} = 0 \tag{A3}$$

$$L = L_{-\infty} \Rightarrow s_1 = s_{R0}$$

Equations (A2) and (A3) imply that when $L = L_{-\infty}$, the steady effluxes of mass and entropy from $\mathscr{L}$ are balanced by the integrals of vanishingly small fluxes of these quantities over surface σ_+ of $\mathscr{L}$. The entrained mass and entropy in $\mathscr{L}$ are each infinite, and remain steady over the specified time interval. The momentum entrained in $\mathscr{L}$ is zero over the specified time interval in this case. Equation (A3) shows that the flow entering the head of the rarefaction wave at plane 0-0 is carrying entropy s_{R0} for $0 \le t < \infty$; and hence that $s = s_{R0}$ throughout domains $\mathscr{L}$ and $\mathscr{D}$, and their intersection, which is plane 0-0.

Let us apply the First Law of Thermodynamics to the expansion from the infinite reservoir into the main. Recall that h defines total, or stagnation enthalpy. The flow is quiescent in the reservoir, and time-steady at plane 0-0 for $0 \le t < \infty$. Now the total enthalpy efflux through the surface, σ, of $\mathscr{L}$, written as: $\iint_{\sigma} (\rho \vec{V}\cdot\hat{n})\cdot h\, d\sigma$, is the sum of two components; these components being expressed as Equations (A4.i) and (A4.ii):

$$\iint_{\sigma+} (\rho \vec{V}\cdot\hat{n})\cdot h\, d\sigma = \iint_{\sigma+} (\rho \vec{V}\cdot\hat{n})\cdot e\, d\sigma + \iint_{\sigma+} (\rho \vec{V}\cdot\hat{n})\cdot (pv)\, d\sigma \tag{A4.i}$$

$$\iint_{\sigma_{0\text{-}0}} (\rho \vec{V}\cdot\hat{n})\cdot h\, d\sigma = \rho_1 U_{cr} \cdot e_1 \cdot \sigma_{0\text{-}0} + \rho_1 U_{cr} \cdot p_1 v_1 \cdot \sigma_{0\text{-}0} \tag{A4.ii}$$

Suppose that L approaches $L_{-\infty}$. The infinite reservoir is in a quiescent state, so:

$$\frac{\partial E_{RO}}{\partial t} = \iiint_{\mathscr{L}} \rho \cdot \frac{De}{Dt} d\tau - \iint_{\sigma+} (\rho \vec{V} \cdot \hat{n}) \cdot e d\sigma - \rho_1 U_{cr} e_1 \cdot \sigma_{0-0} = 0 \tag{A5}$$

$$L = L_{-\infty} \Rightarrow \left\{ \vec{V}(\sigma_+) = 0 \wedge \rho(\sigma_+) = \rho_{RO} \wedge e(\sigma_+) = e_{RO} \right\} for\ \rho_1 e_1 U_{cr} \cdot \sigma_{0-0} > 0$$

Consider a point p in $\mathscr{L}$. In view of the quiescent state of the reservoir:

$$p \in \mathscr{L} \Rightarrow \frac{De}{Dt}(p) = 0 \tag{A6}$$

It follows that when $L=L_{-\infty}$, we should write:

$$\iint_{\sigma+} (\rho \vec{V} \cdot \hat{n}) \cdot e d\sigma = - \rho_1 U_{cr} e_1 \cdot \sigma_{0-0} \tag{A7}$$

It is apparent that when $L=L_{-\infty}$, the efflux of total energy at plane 0-0 no longer materially changes the total energy of the gas contained in the reservoir. Additionally, Equation (A2) implies:

$$\iint_{\sigma+} \rho \vec{V} \cdot \hat{n} d\sigma = -\rho_1 U_{cr} \cdot \sigma_{0-0} \tag{A8}$$

Consequently, Equations (A4.i) an (A4.ii) may be written in the following form:

$$\iint_{\sigma+} (\rho \vec{V} \cdot \hat{n}) \cdot h d\sigma = - \rho_1 U_{cr} \cdot e_1 \cdot \sigma_{0-0} + \iint_{\sigma+} (\rho \vec{V} \cdot \hat{n}) \cdot \left(p_{RO} \cdot v_{RO} \right) d\sigma \tag{A9.i}$$

$$\iint_{\sigma_{0-0}} (\rho \vec{V} \cdot \hat{n}) \cdot h d\sigma = \rho_1 U_{cr} \cdot e_1 \cdot \sigma_{0-0} + \rho_1 U_{cr} \cdot p_1 \cdot v_1 \cdot \sigma_{0-0} \tag{A9.ii}$$

Equations (A9.i) and (A9.ii) imply that when $L=L_{-\infty}$, the integral of the stagnation enthalpy over the surface of $\mathscr{L}$ becomes:

$$\iint_{\sigma} (\rho \vec{V} \cdot \hat{n}) \cdot h d\sigma = \rho_1 U_{cr} \cdot \left(p_1 \cdot v_1 - p_{RO} \cdot v_{RO} \right) \cdot \sigma_{0-0} < 0 \tag{A10}$$

The right-hand side of Inequality (A10) is clearly recognizable as the flow work rate which must be supplied by the reservoir to accelerate the flow from its quiescent state interior to the reservoir to its critical state at plane 0-0. This term must invariably be strictly less than zero. Evidently, the First Law Thermodynamics does not require the total enthalpy efflux from $\mathscr{L}$ to vanish in case the reservoir is infinite in extent, even when the flow at plane 0-0 is fixed in time.

It is clear that a jump discontinuity in the total enthalpy at plane 0-0 is permitted, since the infinite reservoir maintains an inert state of total energy in spite of the facts that it is adiabatic, the flow conditions are time-steady, there is no influx of total energy into the reservoir; but there is an efflux of total energy at plane 0-0. An expression admitting a jump of finite magnitude is given by Inequality (A10). However, the magnitude of the jump is not determined by this inequality. Rather, given that the thermodynamic process is reversible, it is determined by the entropy conservation relation of Equation(A3). One can see that the infinite, one-dimensional reservoir permits an isentropic expansion shock at plane 0-0 which satisfies both the First and Second Laws of Thermodynamics.

Next, we consider the isentropic process at plane 0-0. Let the following definitions hold:

$$r = \frac{P_{RO}}{P_1} \tag{A11}$$

$$\beta_* = \frac{T_{RO}}{T_1} \tag{A12}$$

In view of the application of the Perfect Equation of State, we may write:

$$\rho_1 = \frac{p_1}{RT_1} \tag{A13}$$

Hence, Equation (A1) implies that:

$$p_{RO} - \left(p_1 + \frac{p_1}{RT_1} \cdot U_{cr}^2 \right) = 0 \tag{A14}$$

and we may write:

$$r = 1 + \frac{U_{cr}^2}{RT_1} \tag{A15}$$

Continuing, let us compare β_* with the critical temperature ratio at plane 0-0. it is apparent that the isentropic condition on domains $\mathscr{L}$ and $\mathscr{D}$, as well as their intersection, implies that one may define a function $\beta(z)$ such that:

$$\beta(z) = (z)^{\frac{\gamma-1}{\gamma}} \tag{A16}$$

where z is an arbitrary pressure ratio which is infinitely differentiable. Now, by Taylor's Theorem, there must be a real number ζ between 1 and r such that the following relation must hold:

$$\frac{T_{RO}}{T_1} = \beta(1) + \frac{\beta'(1)}{1!}\cdot(r-1) + \frac{\beta''(1)}{2!}\cdot(r-1)^2 + \frac{\beta'''(\zeta)}{3!}\cdot(r-1)^3 \tag{A17}$$

When Equation (A15) is introduced, Equation (A17) becomes:

$$\frac{T_{RO}}{T_1} = 1 + \frac{(\gamma-1)}{1!}\cdot\frac{U_{cr}^2}{\gamma RT_1} - \frac{(\gamma-1)}{2!}\cdot\left(\frac{U_{cr}^2}{\gamma RT_1}\right)^2 +$$

$$\frac{(\gamma-1)\cdot(\gamma+1)}{3!}\cdot\left(\frac{1}{\zeta}\right)^{\frac{2\gamma+1}{\gamma}}\cdot\left(\frac{U_{cr}^2}{\gamma RT_1}\right)^3 \qquad 1 < \zeta < r \tag{A18}$$

For critical flow at plane 0-0, it must be that:

$$\frac{U_{cr}^2}{\gamma RT_1} = 1 \tag{A19}$$

In this case, Equation (A18) reduces to:

$$\frac{T_{RO}}{T_1} = \frac{(\gamma+1)}{2} + \frac{(\gamma-1)\cdot(\gamma+1)}{6}\cdot\left(\frac{1}{\zeta}\right)^{\frac{2\gamma+1}{\gamma}} \qquad 1 < \zeta < (\gamma+1) \tag{A20}$$

Equation (A20) shows that ratio of reservoir temperature to the temperature at plane 0-0 differs from critical temperature ratio only by the remainder of the Taylor's Expansion after the first three terms. Re-introducing the notation of Equation (A12), we may write an inequality which expresses the difference between the reservoir temperature ratio and critical temperature ratio at plane 0-0. This inequality is:

$$\frac{(\gamma-1)\cdot(\gamma+1)}{6}\cdot\left(\frac{1}{\gamma+1}\right)^{\frac{2\gamma+1}{\gamma}} < \left\{\beta_{*} - \frac{(\gamma+1)}{2}\right\} < \frac{(\gamma-1)\cdot(\gamma+1)}{6} \qquad \textbf{(A21)}$$

Clearly, T_{RO} is steady for $0 \le t < \infty$, and ζ is a unique number. It follows from Equation (A20) that β_{*} is fixed for $0 \le t < \infty$ as well.

Note that if T_{0} symbolizes the stagnation temperature at plane 0-0, and T_{1} symbolizes the static temperature there, when the flow velocity is critical, then:

$$\frac{T_0}{T_1} = \frac{\gamma+1}{2} \qquad \textbf{(A22)}$$

Equation (A22) is the aforementioned critical temperature ratio. Obviously, T_{0} is fixed, and $T_{RO} > T_{0}$ for $0 \le t < \infty$. If the working fluid is standard air, then the critical temperature ratio at plane 0-0 may be calculated as:

$$\frac{T_0}{T_1} = \frac{\gamma+1}{2} = 1.200 \qquad \textbf{(A23)}$$

However, Inequality (A21) implies that:

$$0.01486 < \left\{\beta_{*} - \frac{\gamma+1)}{2}\right\} < 0.16000 \qquad \textbf{(A24)}$$

Clearly, the difference is always greater than zero.

The values of reservoir stagnation pressure and temperature which must be maintained upstream of the expansion shock in order to sustain critical flow at plane 0-0 can be directly calculated from the conditions imposed by Equations (A1), (A2), and (A3). They are given by

Equations (A25) and (A26):

$$p_{RO} = p_1 \cdot \left(1 + \frac{U_{cr}^2}{RT_1} \right) = p_1 \cdot (\gamma + 1) \tag{A25}$$

$$T_{RO} = T_1 \cdot \left(1 + \frac{U_{cr}^2}{RT_1} \right)^{\frac{\gamma - 1}{\gamma}} = T_1 \cdot (\gamma + 1)^{\frac{\gamma - 1}{\gamma}} \tag{A26}$$